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# Asymptotic behavior of solutions for nonlinear elliptic problems with the fractional Laplacian

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ABSTRACT

In this paper we study the asymptotic behavior of least energy solutions and the existence of multiple bubbling solutions of nonlinear elliptic equations involving the fractional Laplacians and the critical exponents. This work can be seen as a nonlocal analog of the results of Han (1991) [24] and Rey (1990) [35].


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## 1. Introduction

The aim of this paper is to study the nonlocal equations:

$$\begin{cases} \mathcal{A}_s u = u^p + \epsilon u & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $0 < s < 1$, $p := \frac{n+2s}{n-2s}$, $\epsilon > 0$ is a small parameter, $\Omega$ is a smooth bounded domain of $\mathbb{R}^n$ and $\mathcal{A}_s$ denotes the fractional Laplace operator $(-\Delta)^s$ in $\Omega$ with zero Dirichlet boundary values on $\partial\Omega$, defined in terms of the spectra of the Dirichlet Laplacian $-\Delta$ on $\Omega$. It can be understood as the nonlocal version of the Brezis–Nirenberg problem [3].

The fractional Laplacian appears in diverse areas including physics, biological modeling and mathematical finances and partial differential equations involving the fractional Laplacian have attracted the attention of many researchers. An important feature of the fractional Laplacian is its nonlocal property, which makes it difficult to handle. Recently, Caffarelli and Silvestre [7] developed a local interpretation of the fractional Laplacian given in $\mathbb{R}^n$ by considering a Neumann type operator in the extended domain $\mathbb{R}^{n+1}_+ := \{(x,t) \in \mathbb{R}^{n+1}\colon t > 0\}$. This observation made a significant influence on the study of related nonlocal problems. A similar extension was devised by Cabré and Tan [6] and Capella, Dávila, Dupaigne, and Sire [8] (see Brändle, Colorado, de Pablo, and Sánchez [2] and Tan [40] also) for nonlocal elliptic equations on bounded domains with the zero Dirichlet boundary condition, and by Kim and Lee [25] for singular nonlocal parabolic equations.

Based on these extensions, many authors studied nonlinear problems of the form $\mathcal{A}_s u = f(u)$, where $f : \mathbb{R}^n \to \mathbb{R}$ is a certain function. Since it is almost impossible to describe all the works involving them, we explain only some results which are largely related to our problem. When $s = \frac{1}{2}$, Cabré and Tan [6] established the existence of positive solutions for equations having nonlinearities with the subcritical growth, their regularity and the symmetric property. They also proved a priori estimates of the Gidas–Spruck type by employing a blow-up argument along with a Liouville type result for the square root of the Laplacian in the half-space. Moreover, Tan [39] studied Brezis–Nirenberg type problems (see [3]) for the case $s = \frac{1}{2}$, that is, when the nonlinearity is given by $f(u) = u^{\frac{n+1}{n-1}} + \epsilon u$ with $\epsilon > 0$. On the other hand, in [12], the first author of this paper gave a different proof for the Gidas–Spruck type estimates using the Pohozaev identity and applied them to the Lane–Emden type system involving $A_{1/2}$. The work of Tan [39] is extended to $0 < s < 1$ and $f(u) = u^{\frac{n+2s}{n-2s}} + \lambda u^q$ for $0 < q < \frac{n+2s}{n-2s}$ in [1]. See also [2] which dealt with a subcritical concave-convex problem. For $f(u) = u^q$ with the critical and supercritical exponents $q \geqslant \frac{n+2s}{n-2s}$, the nonexistence of solutions was proved in [2,39,40] in which the authors devised and used the Pohozaev type identities.