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Decompositions and complexifications of some infinite-dimensional homogeneous spaces

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ABSTRACT

In this paper an extended Corach–Porta–Recht decomposition theorem for Finsler symmetric spaces of semi-negative curvature in the context of reductive structures is proven. This decomposition theorem is applied to give a geometric description of the complexification of some infinite dimensional homogeneous spaces.

1. Introduction

In recent years, the geometrical study of operator algebras and their homogeneous spaces has become a central topic in the study of infinite dimensional geometry. It is a source of examples and counterexamples, and the operator algebra techniques (Banach

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algebras and C^* algebras, with their distinguished tools) are being used for obtaining results on abstracts infinite dimensional manifolds by studying their groups of automorphism, isometries, and their associated fiber bundles and G-bundles. See the recent book [1] by D. Beltita for a full account of these objects and a comprehensive list of references.

In particular, what we are interested in here, is the extension of certain results on the geometric description of complexifications of homogeneous spaces of Banach–Lie groups studied by Beltita and Galé in [2] and also the decompositions of the acting groups by means of a series of chained reductive structures.

In Section 2 the reader can find the basic facts about Finsler symmetric spaces; these are spaces of the form G/U endowed with a Finsler structure, where G is a Banach–Lie group and U is the fixed point set of an involution σ on G. A criteria that ensures that the spaces G/U have semi-negative curvature is recalled from the work of Neeb [12].

In Section 3 we recall the definition of reductive structures, which can be interpreted as connection forms E on homogeneous spaces of the form G_A/G_B. Examples in the context of operator algebras are given: conditional expectations, their restrictions to Schatten ideals and projections to corners of operator algebras. The Corach–Porta–Recht splitting theorem by Conde and Larotonda [5] is used to prove an extended Corach–Porta–Recht splitting theorem in the context of several reductive structures.

In Section 4 the Corach–Porta–Recht splitting theorem is used to give a geometric description of homogeneous spaces of the form G_A/G_B as associated principal bundles over U_A/U_B. Under additional hypothesis about the holomorphic character of G_A and the involution σ on G_A it is possible to interpret G_A/G_B as the complexification of U_A/U_B. Under these additional assumptions G_A/G_B is identified with the tangent bundle of U_A/U_B and it is shown that this identification has nice functorial properties related to the connection form E. Finally, we use the three examples of connection forms introduced in Section 3, to give a geometrical description of the complexifications of flag manifolds, coadjoint orbits in Schatten ideals and Stiefel manifolds respectively.

2. Finsler symmetric spaces

A connected Banach–Lie group G with an involutive automorphism σ is called a *symmetric Banach–Lie group*. Let $\mathfrak{g}$ be the Banach–Lie algebra of G, and let $U = \{g \in G\colon\ \sigma(g) = g\}$ be the subgroup of fixed points of σ. Then the Banach–Lie algebra $\mathfrak{u}$ of U is a closed and complemented subspace of $\mathfrak{g}$; a complement is given by the closed subspace

$$\mathfrak{p} = \{X \in \mathfrak{g}\colon\ \sigma_{*1}X = -X\},$$

where for a smooth map between manifolds $f : X \to Y$ we use the notation $f_* : T(X) \to T(Y)$ for the tangent map and $f_{*x} : T_x(X) \to T_{f(x)}(Y)$ for the tangent map at $x \in X$.

The Lie algebra $\mathfrak{u}$ is the eigenspace of σ_{*1} corresponding to the eigenvalue $+1$ and $\mathfrak{p}$ is the eigenspace corresponding to the eigenvalue -1. Since $\mathfrak{u}$ is complemented U is