Available online at www.sciencedirect.com

SciVerse ScienceDirect

Journal of Functional Analysis 265 (2013) 953–982

JOURNAL OF Functional Analysis

www.elsevier.com/locate/jfa

Blow-up and nonlinear instability for the magnetic Zakharov system ☆

Zaihui Gan [a,c], Boling Guo [b], Daiwen Huang [b,*]

[a] College of Mathematics and Software Science, Sichuan Normal University, Chengdu 610068, China
[b] Institute of Applied Physics and Computational Mathematics, Beijing 100088, China
[c] Center for Applied Mathematics, Tianjin University, Tianjin 300072, China

Received 26 August 2012; accepted 4 May 2013

Available online 27 May 2013

Communicated by F.-H. Lin

Abstract

This study deals with the generalized Zakharov system with magnetic field. First of all, we construct a kind of blow-up solutions and establish the existence of blow-up solutions to the system through considering an elliptic system. Next, we show the nonlinear instability for a kind of periodic solutions. In addition, we consider the concentration properties of blow-up solutions for the system under study. At the end of this paper, we establish the global existence of weak solutions to the Cauchy problem of the system under consideration.

Keywords: Generalized Zakharov system; Blow-up solutions; Nonlinear instability; Concentration properties; Magnetic field

☆ This work is supported by 973 Program (grant No. 2013CB834100), National Natural Science Foundation of PR China (11171241, 11071177, 91130005, 11271052) and Program for New Century Excellent Talents in University (NCET 12-1058).

* Corresponding author.
E-mail address: hdw55@tom.com (D. Huang).

0022-1236/$ – see front matter
http://dx.doi.org/10.1016/j.jfa.2013.05.017

1. Introduction

In this paper, we study the Cauchy problem of a generalized Zakharov system with magnetic field:

$$\begin{cases} i\mathbf{E}_t + \triangle\mathbf{E} - n\mathbf{E} + i(\mathbf{E}\wedge\mathbf{B}) = 0, \\ \frac{1}{c_0^2}n_{tt} - \triangle n = \triangle|\mathbf{E}|^2, \\ \triangle\mathbf{B} - i\eta\,\nabla\times\big(\nabla\times(\mathbf{E}\wedge\bar{\mathbf{E}})\big) + \beta\mathbf{B} = 0, \end{cases} \tag{1.1}$$

$$\mathbf{E}(0,x) = \mathbf{E}_0(x), \qquad n(0,x) = n_0(x), \qquad n_t(0,x) = n_1(x), \tag{1.2}$$

where $\mathbf{E}(t,x)$ is a vector-valued function from $\mathbb{R}^+\times\mathbb{R}^2$ into $\mathbb{C}^3$ and denotes the slowly varying complex amplitude of the high-frequency electric field, $n(t,x)$ is a function from $\mathbb{R}^+\times\mathbb{R}^2$ into $\mathbb{R}$ and represents the fluctuation of the electron density from its equilibrium, the self-generated magnetic field $\mathbf{B}$ is a vector-valued function from $\mathbb{R}^+\times\mathbb{R}^2$ into $\mathbb{R}^3$, $i^2=-1$, constants $\eta>0$, $\beta\leqslant 0$, $\bar{\mathbf{E}}$ is the complex conjugate of $\mathbf{E}$, and $\wedge$ means the exterior product of vector-valued functions. System (1.1) describes the spontaneous generation of a magnetic field in a cold plasma (see Ref. [8] for the physical derivation).

If we neglect the magnetic field, system (1.1) reduces the following classical Zakharov system:

$$\begin{cases} i\mathbf{E}_t + \triangle\mathbf{E} - n\mathbf{E} = 0, \\ \frac{1}{c_0^2}n_{tt} - \triangle n = \triangle|\mathbf{E}|^2, \end{cases} \tag{ZS}$$

which describes the propagation of Langmuir waves (cf. [17]). There are many papers concerning the well-posedness of the Zakharov system (ZS) (see, e.g., [1,4,5,3,12–14] and references therein). On this topic, for (1.1) there are also some works (cf. [2,6,7,10,18]).

Let $\mathbf{E}=(E_1,E_2,0)$, $\mathbf{B}=-i\eta\mathcal{F}^{-1}(\frac{|\xi|^2}{|\xi|^2-\beta}\mathcal{F}(\mathbf{E}\wedge\bar{\mathbf{E}}))$, $E_1(t,x),\ E_2(t,x)\in\mathbb{C}$, $x\in\mathbb{R}^2$. For $n_1\in H^{-1}$, there exist $\omega_0\in L^2(\mathbb{R}^2)$ and $\mathbf{v}_0\in L^2(\mathbb{R}^2)$ such that $n_t(0,x)=n_1=-\operatorname{div}\mathbf{v}_0+w_0$. In this case, (1.1)–(1.2) can be rewritten as follows:

$$\begin{cases} i\mathbf{E}_t + \triangle\mathbf{E} - n\mathbf{E} + i\big(\mathbf{E}\wedge\mathbf{B}(\mathbf{E})\big) = 0, \\ n_t = -\operatorname{div}\mathbf{v} + w_0, \\ \frac{1}{c_0^2}\mathbf{v}_t = -\nabla\big(n+|E|^2\big), \\ \mathbf{E}(0,x) = \mathbf{E}_0(x), \quad n(0,x) = n_0(x), \quad \mathbf{v}(0,x) = \mathbf{v}_0(x). \end{cases} \tag{1.3}$$

In the present paper, we first study the existence of blow-up solutions for the Cauchy problem (1.1)–(1.2). We construct a kind of blow-up solutions to (1.1)–(1.2) on $[0,T)$, which has the form:

$$\mathbf{E} = (E_1, -iE_1, 0), \qquad n(t,x) = \frac{\omega^2}{(T-t)^2}\tilde{N}\left(\frac{x\omega}{T-t}\right), \tag{1.4}$$

where