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On Liouville systems at critical parameters, Part 1: One bubble

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Abstract

In this paper we consider bubbling solutions to the general Liouville system:

$$\Delta_g u_i^k + \sum_{j=1}^n a_{ij}\rho_j^k\left(\frac{h_j e^{u_j^k}}{\int h_j e^{u_j^k}} - 1\right) = 0 \quad \text{in } M,\ i = 1, \dots, n\ (n \geqslant 2) \tag{0.1}$$

where (M, g) is a Riemann surface, and $A = (a_{ij})_{n\times n}$ is a constant non-negative matrix and $\rho_j^k \to \rho_j$ as $k \to \infty$. Among other things we prove the following sharp estimates.

(1) The location of the blowup point.
(2) The convergence rate of $\rho_j^k - \rho_j$, $j = 1, \dots, n$.

These results are of fundamental importance for constructing bubbling solutions. It is interesting to compare the difference between the general Liouville system and the $SU(3)$ Toda system on estimates (1) and (2).
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1. Introduction

Let (M,g) be a compact Riemann surface whose volume is normalized to be 1, $h_1,\dots,h_n$ be positive C^3 functions on M, $\rho_1,\dots,\rho_n$ be non-negative constants. In this article we continue our study of the following Liouville system defined on (M,g):

$$\Delta_g u_i+\sum_{j=1}^{n}\rho_j a_{ij}\left(\frac{h_j e^{u_j}}{\int_M h_j e^{u_j}\,dV_g}-1\right)=0,\quad i\in I:=\{1,\dots,n\} \tag{1.1}$$

where dV_g is the volume form, $A=(a_{ij})$ is a non-negative constant matrix, Δ_g is the Laplace–Beltrami operator ($-\Delta_g\geqslant 0$). When $n=1$ and $a_{11}=1$, Eq. (1.1) is the mean field equation of the Liouville type:

$$\Delta_g u+\rho\left(\frac{he^u}{\int_M he^u\,dV_g}-1\right)=0\quad \text{in } M. \tag{1.2}$$

Therefore, the Liouville system (1.1) is a natural extension of the classical Liouville equation, which has been extensively studied for the past three decades. Both the Liouville equation and the Liouville system are related to various fields of geometry, Physics, Chemistry and Ecology. For example in conformal geometry, when $\rho=8\pi$ and M is the sphere $\mathbb{S}^2$, Eq. (1.2) is equivalent to the famous Nirenberg problem. For a bounded domain in $\mathbb{R}^2$ and $n=1$, a variant of (1.2) can be derived from the mean field limit of Euler flows or spherical Onsager vortex theory, as studied by Caglioti, Lions, Marchioro and Pulvirenti [6,7], Kiessling [27], Chanillo and Kiessling [9] and Lin [33]. In classical gauge field theory, Eq. (1.1) is closely related to the Chern–Simons–Higgs equation for the abelian case, see [5,23,24,44]. Various Liouville systems are also used to describe models in the theory of self-gravitating systems [1], Chemotaxis [16,26], in the physics of charged particle beams [4,19,28,29], in the non-abelian Chern–Simons–Higgs theory [20,25,44] and other gauge field models [21,22,30]. For recent developments of these subjects or related Liouville systems in more general settings, we refer the readers to [2,3,10–15,17,18,31–36,39–43, 45,46] and the references therein.

For any solution u of (1.2), clearly adding any constant to u gives another solution. So it is nature to assume $u\in \mathring{H}^1(M)$, where

$$\mathring{H}^1(M)=\left\{u\in L^2(M)\;\middle|\;|\nabla_g u|\in L^2(M)\text{ and }\int_M u\,dV_g=0\right\}.$$

Corresponding to (1.1) we set

$$\mathring{H}^{1,n}=\mathring{H}^1(M)\times\cdots\times\mathring{H}^1(M)$$

to be the space for solutions. For any $\rho=(\rho_1,\dots,\rho_n)$, $\rho_i>0$ ($i\in I=\{1,\dots,n\}$), let Φ_ρ be a nonlinear functional defined in $\mathring{H}^{1,n}$ by