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## Noncommutative polynomials nonnegative on a variety intersect a convex set



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### ABSTRACT

This paper gives a precise algebraic certificate for a noncommutative polynomial to be nonnegative on a free convex semialgebraic set intersect the variety of a free left ideal.

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