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A theorem on measures in dimension 2 and applications to vortex sheets

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ABSTRACT

We find conditions under which measures belong to $H^{-1}(\mathbb{R}^2)$. Next we show that measures generated by the Prandtl, Kaden as well as Pullin spirals, objects considered by physicists as incompressible flows generating vorticity, satisfy assumptions of our theorem, thus they are (locally) elements of $H^{-1}(\mathbb{R}^2)$. Moreover, as a by-product, we prove an embedding of the space of Morrey type measures in H^{-1}.

1. Introduction

Due to d'Alembert's paradox we know that in the class of regular steady irrotational Euler flows the lift exerted by the inviscid incompressible flow on a three-dimensional body is zero, see [10, Appendix 1.4]. On the other hand, among regular solutions to the

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Euler equation, initially potential flow remains potential during the evolution, according to the Helmholtz theorem. Hence, physicists (not only) were looking for the irregular flows generating the vorticity.

It was Felix Klein, in his famous Kaffelöffel experiment, who postulated that the flow in which all the vorticity is supported on a spiral may generate the vorticity [6], [16, Chapter 6.1]. Such spirals are examples of *vortex sheets*, i.e. flows whose vorticity is supported on a curve, being zero off the curve.

The main theoretical efforts in constructing candidates for irregular solutions to the 2d Euler equations were made in the school of Prandtl. The best known examples of vortex sheets are self-similar spirals introduced by Prandtl, see [12], Kaden, see [4], or Anton, see [1]. A broader class of such self-similar spirals come from the numerical experiments carried by Pullin [13].

Self-similar spirals of vorticity were observed experimentally and obtained by theoretical considerations, see Section 4, but it is still not known if, and in what sense, such objects can be interpreted as solutions to the 2d incompressible Euler equation. Notice that even if some examples of flows generating the vorticity do not solve the 2d Euler equation they can still bring some important information, being for instance solutions to Euler-like problems. A well known example of such a phenomenon is the non-spiral Prandtl–Munk steady vortex sheet. It was shown in [8] that it is actually a weak solution to the nonhomogeneous Euler equation with forcing term representing a tension force applied to the tips of a wing.

We give an answer to a question whether spirals of vorticity are locally in $H^{-1}(\mathbb{R}^2)$, i.e. whether their restriction to any compact set belongs to $H^{-1}(\mathbb{R}^2)$. There are several reasons for studying the above question. The framework of weak solutions to the 2d Euler equations capturing the vortex sheets introduced in the pioneering work [3] consists of a few requirements concerning the velocity field v in order to interpret it as a vortex sheet solution to the 2d Euler equations. One of them is the following

$$v \in L^\infty\big(0, T; L^2_{loc}\big(\mathbb{R}^2\big)\big). \tag{1.1}$$

It is not only the technical assumption, so that weak velocity formulation makes sense, but also the physical interpretation of (1.1) is clear; kinetic energy of the fluid is finite at least locally. Our question if the measure supported on the spiral of vorticity belongs locally to H^{-1} is clearly related to (1.1).

Delort proved the global existence of vortex sheet type weak solutions to the 2d Euler equation [2]. The assumptions he imposes are that the initial vorticity is a compactly supported Radon measure with a fixed sign (positive or negative) being an element of H^{-1}. Hence if one wants to see whether flows generating vorticity can be understood as Delort's solutions, one has to check if they initially belong to H^{-1}. Delort's theorem concerns only compactly supported data, yet one may hope to extend the theorem to the non-compact case.