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Asymptotic mean value formula for sub-p-harmonic functions on the Heisenberg group ☆

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Abstract

In this paper, we prove an asymptotic mean value formula of sub-p-harmonic functions in the viscosity sense on the Heisenberg group. As an application, we give a new proof of the Harnack inequality for sub-p-harmonic functions on the Heisenberg group.

Keywords: Heisenberg group; Mean value formula; Sub-p-harmonic; Viscosity solution; Taylor expansion

1. Introduction

It is well known that a basic property of harmonic functions is the mean value property [1]. More precisely, u is a harmonic function in a domain $\Omega \subset R^n$ (that is u satisfies $\Delta u = 0$ in Ω) if and only if it satisfies the mean value formula

$$u(x) = \fint_{B_\varepsilon(x)} u(y)\, dy$$

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whenever $B_\varepsilon(x) \subset \Omega$ and $⨍_E f$ denotes the average of f over the set E. In fact, we can relax this condition by requiring that it holds asymptotically

$$u(x) = ⨍_{B_\varepsilon(x)} u(y)\, dy + o\big(\varepsilon^2\big) \quad \text{as } \varepsilon \to 0.$$

This result follows for C^2 functions by using the Taylor expansion and for continuous functions by using the theory of viscosity solutions. In addition, an asymptotic mean value formula holds for some nonlinear cases as well. Manfredi et al. [2] characterized p-harmonic functions by means of asymptotic mean value properties that hold in the so-called viscosity sense (see Definition 2.2 below). More precisely, they proved that the asymptotic expansion

$$u(x) = \frac{\alpha}{2}\Big\{\max_{\overline{B_\varepsilon(x)}} u + \min_{\overline{B_\varepsilon(x)}} u\Big\} + \beta ⨍_{B_\varepsilon(x)} u(y)\, dy + o\big(\varepsilon^2\big)$$

as $\varepsilon \to 0$ holds in the viscosity sense for all $x \in \Omega$ if and only if

$$-\Delta_p u = -div\big(|\nabla u|^{p-2}\nabla u\big) = 0$$

in Ω in the viscosity sense, where the constants α and β are given by

$$\alpha = \frac{p-2}{p+n} \quad \text{and} \quad \beta = \frac{2+n}{p+n}.$$

The goal of this paper is to extend this analysis to sub-elliptic cases and to study asymptotic mean value properties of the sub-p-harmonic functions on the Heisenberg group.

Let us introduce the Heisenberg group firstly. The Heisenberg group H^n is a nilpotent Lie group of step two whose underlying manifold is $R^{2n} \times R$ with coordinates $(z,t) = (x,y,t) = (x_1,\dots,x_n,y_1,\dots,y_n,t)$ and whose group action $\circ$ is given by

$$(x_0,y_0,t_0)\circ(x,y,t) = \left(x + x_0, y + y_0, t + t_0 + 2\sum_{i=1}^n (x_i y_{0_i} - y_i x_{0_i})\right). \tag{1}$$

A basis for the Lie algebra of left-invariant vector fields on H^n is given by

$$\begin{cases} X_i = \dfrac{\partial}{\partial x_i} + 2y_i\dfrac{\partial}{\partial t}, & i = 1,\dots,n,\\ X_{n+i} = \dfrac{\partial}{\partial y_i} - 2x_i\dfrac{\partial}{\partial t}, & i = 1,\dots,n,\\ T = \dfrac{\partial}{\partial t}. \end{cases} \tag{2}$$

From (2), it is easy to check that X_i and X_{n+j} satisfy

$$[X_i, X_{n+j}] = -4T\delta_{ij}, \quad \text{and} \quad [X_i, X_j] = [X_{n+i}, X_{n+j}] = 0, \quad i,j = 1,\dots,n.$$