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Global Div-Curl lemma in negative Sobolev spaces

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Abstract

This paper studies the global version of the Div-Curl lemma in negative Sobolev spaces in bounded domains in $\mathbb{R}^3$. We proved that for the two sequences $\{\mathbf{u}_j\}_{j=1}^{\infty}$ and $\{\mathbf{v}_j\}_{j=1}^{\infty}$ converging to $\mathbf{u}$ and $\mathbf{v}$ weakly in $L^r(\Omega)$ and $L^{r'}(\Omega)$, respectively, with r and r' being conjugate exponents and $1 < r < \infty$, suppose that $\{\operatorname{curl} \mathbf{v}_j\}_{j=1}^{\infty}$ is compact in the dual space of $W^{1,r}(\Omega)$, and also if $\{\operatorname{div} \mathbf{u}_j\}_{j=1}^{\infty}$ and the normal part of $\mathbf{u}_j$ on boundary are compact in the dual of $W_0^{1,r'}(\Omega)$ and in $W^{-1/r,r}(\partial\Omega)$ respectively or $\{\operatorname{div} \mathbf{u}_j\}_{j=1}^{\infty}$ and the tangential part of $\mathbf{v}_j$ on boundary are compact in the dual of $W^{1,r'}(\Omega)$ and in $W^{-1/r',r'}(\partial\Omega)$ respectively, then we have $\int_\Omega \mathbf{u}_j \cdot \mathbf{v}_j \, dx \to \int_\Omega \mathbf{u} \cdot \mathbf{v} \, dx$ provided $\operatorname{div} \mathbf{u}_j \in L^r$ and $\operatorname{curl} \mathbf{v}_j \in L^{r'}$. As an application, the classical Div-Curl lemma will be given another proof.

Keywords: Div-Curl lemma; Helmholtz–Weyl decomposition; Negative indices; Bounded domains

1. Introduction

The classical Div-Curl lemma developed by Murat [13] and Tartar [18–20] is an important part of the theory of compensated compactness. It states as follows: Suppose that the vector sequences $\{\mathbf{u}_j\}_{j=1}^{\infty}$ and $\{\mathbf{v}_j\}_{j=1}^{\infty}$ converge weakly in $L^2(\Omega)$ to $\mathbf{u}$ and $\mathbf{v}$ respectively, and if $\{\operatorname{div} \mathbf{u}_j\}_{j=1}^{\infty}$ and $\{\operatorname{curl} \mathbf{v}_j\}_{j=1}^{\infty}$ are both compact in $H_{\mathrm{loc}}^{-1}(\Omega)$, then

$$\mathbf{u}_j \cdot \mathbf{v}_j \to \mathbf{u} \cdot \mathbf{v} \quad \text{in } \mathcal{D}'(\Omega). \tag{1.1}$$

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For the Div-Curl lemma in L^r spaces with $1 < r < \infty$, Murat in [14] showed that the inner product of two bounded sequences $\{\mathbf{u}_j\}_{j=1}^{\infty}$ and $\{\mathbf{v}_j\}_{j=1}^{\infty}$ in $L^r(\Omega)$ and $L^{r'}(\Omega)$, respectively, is weakly continuous, provided $\{\operatorname{div}\mathbf{u}_j\}_{j=1}^{\infty}$ is compact in $W^{-1,r}(\Omega)$ and $\{\operatorname{curl}\mathbf{v}_j\}_{j=1}^{\infty}$ is compact in $W^{-1,r'}(\Omega)$, where $1/r + 1/r' = 1$.

We can see that the convergence (1.1) holds in the sense of distribution, and hence it lacks the description of the convergence up to the boundary. Naturally, one may ask the question of whether we have the continuity for the integral on the whole domain, i.e.

$$\int_{\Omega} \mathbf{u}_j \cdot \mathbf{v}_j \, dx \to \int_{\Omega} \mathbf{u} \cdot \mathbf{v} \, dx \quad \text{as } j \to \infty. \tag{1.2}$$

An immediate consequence of (1.2) is the norm convergence if we have $\mathbf{u}_j = \mathbf{v}_j \in L^2(\Omega)$. This point can't be obtained from the classical Div-Curl lemma.

Using the Helmholtz–Weyl decomposition, Kozono and Yanagisawa in [10] showed that (1.2) holds for the two weakly convergent sequences $\{\mathbf{u}_j\}_{j=1}^{\infty} \in L^r(\Omega)$ and $\{\mathbf{v}_j\}_{j=1}^{\infty} \in L^{r'}(\Omega)$, if we have that

(i) $\{\operatorname{div}\mathbf{u}_j\}_{j=1}^{\infty}$ is bounded in $L^q(\Omega)$ with $q > \max\{1, 3r/(3+r)\}$,
(ii) $\{\operatorname{curl}\mathbf{v}_j\}_{j=1}^{\infty}$ is bounded in $L^s(\Omega)$ with $s > \max\{1, 3r'/(3+r')\}$,
(iii) the normal part of $\mathbf{u}_j$ in $W^{1-1/q,q}(\partial\Omega)$ or the tangential part of $\mathbf{v}_j$ in $W^{1-1/s,s}(\partial\Omega)$ is bounded.

Compared with the consideration in negative Sobolev spaces for the classical Div-Curl lemma, the result obtained by Kozono and Yanagisawa is in the frame of the boundedness of the divergence and the curl of the vector sequences in the usual L^r Sobolev spaces. We know that, by the compact imbedding, there is a gap from L^q boundedness with $q > 3r/(3+r)$ to $W^{-1,r}$ compactness. Therefore, to obtain the convergence (1.2) the weaker conditions that the compactness of the $\operatorname{div}\mathbf{u}_j$ in $W^{-1,r}$ and the $\operatorname{curl}\mathbf{v}_j$ in $W^{-1,r/(r-1)}$ are proposed.

In this paper we examine the convergence (1.2) under the assumption that $\operatorname{div}\mathbf{u}_j$ and $\operatorname{curl}\mathbf{v}_j$ are compact in negative Sobolev spaces, and also we require that the normal part of $\mathbf{u}_j$ or the tangential part of $\mathbf{v}_j$ is compact in some negative Sobolev spaces over the boundary of the domain. It is worth pointing out that the conditions of our result will be strictly weaker than that of Kozono and Yanagisawa's in [10] provided the $\operatorname{div}\mathbf{u}_j$ and the $\operatorname{curl}\mathbf{v}_j$ belong to the spaces L^r and $L^{r/(r-1)}$ respectively, see the discussion in Remark 2.7 below.

The method of our proof is based on the Helmholtz–Weyl decomposition (see [9, Theorem 2.1]):

$$\mathbf{u} = \nabla p_{\mathbf{u}} + \operatorname{curl}\mathbf{w}_{\mathbf{u}} + \mathscr{H}_{\mathbf{u}},$$

where $\mathscr{H}_{\mathbf{u}}$ is the harmonic part, then we get the strong convergence of the gradient part ∇p and the curl part $\operatorname{curl}\mathbf{w}$ associated with the vectors $\mathbf{u}_j$ and the vectors $\mathbf{v}_j$. This technique is the same as that in the proof of [10, Theorem 1] and of [15, Lemma 3.1] for the vectors with compact support. However, the tools we used and the spaces we discussed in this paper are different from Kozono and Yanagisawa's. Indeed, Kozono and Yanagisawa applied the classical $W^{2,r}$ estimate obtained by Agmon, Douglis and Nirenberg in [2] for elliptic systems, together with the variational inequalities involving $\operatorname{curl}\mathbf{w}$ and ∇p, to get the strong convergence for the function p