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Point potential for the generator of a stable process

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Abstract

We consider a closed self-adjoint extension of the infinitesimal generator of a symmetric stable process whose domain core is $C_c^\infty(\mathbf{R^d} \setminus \{\mathbf{0}\})$. Such extensions of the Laplacian have been used in models of the hydrogen atom and more recently have modeled pinning at the origin of polymer models based on Brownian motion. We also outline the construction of pinned polymer models based on the symmetric stable processes, even when the underlying stable process does not possess a local time at the origin.

Keywords: Point potential; Self-adjoint extension; Stable process; Pinned polymer

1. Introduction

The generator of a d-dimensional symmetric stable process of index α acting on $L^2(\mathbf{R^d})$ is a self-adjoint operator which we denote by A. We want to demonstrate several ways to give precise mathematical meaning to the expression $A + \beta$“δ_0”, where δ_0 represents a point potential, or zero-range potential, and β is a real constant. If the dimension $d = 1$ and $1 < \alpha \leqslant 2$, then the existence of local time at 0 makes this fairly routine. However, even when the capacity of the origin is zero, this is also possible when the dimension when $d = \alpha = 1$ or when $d = 2$ or 3 and the index $\alpha \in (\frac{d}{2}, d \wedge 2]$. The operator $\Delta + \beta$“δ_0” with a zero-range potential “δ_0” was introduced

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by Bethe and Peierls in a mathematical model for the hydrogen atom. It was also studied in the context of quantum models in [1]. In [7], it was used to model polymers in $\mathbf{R^d}$ subject to pinning at the origin. The family of operators $\Delta + \beta$"δ_0" arises as the one parameter family of closed self-adjoint extensions of Δ acting on the core $C_c^\infty(\mathbf{R^d} \setminus \{\mathbf{0}\})$. The case of these operators arising from the Laplacian was used in [7] to construct a mathematical model for homopolymers. The appearance of a positive eigenvalue in the spectrum for high values of β was interpreted in terms of a phase transition of the polymer.

Let $\{X_t\colon t \geqslant 0\}$ be a symmetric α-stable process on $\mathbf{R^d}$, defined on a probability space $(\Omega, \{\mathcal{F}_t\}_{t\geqslant 0}, P)$ with characteristic function $\varphi(u) = E[e^{i\langle u, X_1\rangle}]$, $u \in \mathbf{R^d}$. Informative references on these processes can be found in [3] and [4]. If $\alpha = 2$ then $\{X_t\colon t \geqslant 0\}$ is a symmetric Brownian motion, and $\varphi(u) = e^{-\frac{1}{2}\langle u, Cu\rangle}$ where $C = (c_{j,k})$ is a positive semidefinite matrix. If $0 < \alpha < 2$ then

$$\varphi(u) = \exp\left(-\int_S \left|\langle \xi, u\rangle\right|^\alpha \sigma(d\xi)\right)$$

where $S = \{\xi \in \mathbf{R^d}\colon |\xi| = 1\}$ is the unit sphere, and σ is a finite symmetric measure on S (uniquely determined by the distribution of X_1). It is convenient to write $\varphi(u) = \exp(-\upsilon(\eta)|u|^\alpha)$, with $\eta = |u|^{-1}u$ and $\upsilon(\eta)$ the function on the unit sphere uniquely defined by $\varphi(u)$. The relationship of σ with the Lévy measure ν of X_1 is the following: if X_1 is α-stable, $0 < \alpha < 2$, then in polar coordinates ν decomposes into a product measure $\nu(dx) = \lambda(d\xi) r^{-1-\alpha}\, dr$, $x = r\xi$, where λ is a finite measure on the unit sphere and

$$\sigma(d\xi) = \begin{cases} -\Gamma(-\alpha)\cos(\frac{\pi\alpha}{2})\lambda(d\xi), & \text{if } 0 < \alpha < 1, \\ \frac{\pi}{2}\lambda(d\xi), & \text{if } \alpha = 1, \\ -\Gamma(-\alpha)\cos(\frac{\pi\alpha}{2})\lambda(d\xi), & \text{if } 1 < \alpha < 2, \end{cases}$$

where Γ is the usual gamma function.

Consider now the semigroup of operators P_t on $L^2(\mathbf{R^d})$ defined by

$$P_t\psi(x) = E\left[\psi(x + X_t)\right]$$

and its generator

$$A\psi = \lim_{t\to 0} \frac{P_t\psi - \psi}{t} \tag{1}$$

defined on all L^2 functions ψ for which the limit exists. By the symmetry of the process, A is a self-adjoint operator. Its action is best seen in Fourier space where

$$\widehat{A\psi}(u) = -\upsilon(\eta)|u|^\alpha \hat{\psi}(u).$$

Since the dimension d plays an important role in the definition of the point potential, we must avoid the situation where the process $\{X_t\}$ is concentrated on a subspace; we do this by assuming $\varphi(u) = 1$ only if $u = 0$.