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Journal of Functional Analysis 266 (2014) 1565–1610

JOURNAL OF Functional Analysis

www.elsevier.com/locate/jfa

Semigroups of Herz–Schur multipliers

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Received 3 April 2013; accepted 6 November 2013

Available online 15 November 2013

Communicated by S. Vaes

Abstract

In order to investigate the relationship between weak amenability and the Haagerup property for groups, we introduce the weak Haagerup property, and we prove that having this approximation property is equivalent to the existence of a semigroup of Herz–Schur multipliers generated by a proper function (see Theorem 1.2). It is then shown that a (not necessarily proper) generator of a semigroup of Herz–Schur multipliers splits into a positive definite kernel and a conditionally negative definite kernel. We also show that the generator has a particularly pleasant form if and only if the group is amenable. In the second half of the paper we study semigroups of radial Herz–Schur multipliers on free groups. We prove that a generator of such a semigroup is linearly bounded by the word length function (see Theorem 1.6).

Keywords: Herz–Schur multipliers; Approximation properties

1. Introduction

The notion of amenability for groups was introduced by von Neumann [17] and has played an important role in the field of operator algebras for many years. It is well-known that amenability of a group is reflected by approximation properties of the C^*-algebra and von Neumann algebra associated with the group. More precisely, a discrete group is amenable if and only if its (reduced or universal) group C^*-algebra is nuclear if and only if its group von Neumann algebra is semidiscrete.

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[1] Supported by ERC Advanced Grant No. OAFPG 247321 and the Danish National Research Foundation through the Centre for Symmetry and Deformation (DNRF92).

0022-1236/$ – see front matter
http://dx.doi.org/10.1016/j.jfa.2013.11.002

Amenability may be seen as a rather strong condition to impose on a group, and several weakened forms have appeared, two of which are *weak amenability* and the *Haagerup property*. Recall that a discrete group G is amenable if and only if there is a net $(\varphi_i)_{i\in I}$ of finitely supported, positive definite functions on G such that $\varphi_i \to 1$ pointwise. When the discrete group is countable, which will always be our assumption in this paper, we can of course assume that the net is actually a sequence. We have included a few well-known alternative characterizations of amenability in Theorem 5.1.

A countable, discrete group G is called *weakly amenable* if there exist $C > 0$ and a net $(\varphi_i)_{i\in I}$ of finitely supported Herz–Schur multipliers on G converging pointwise to 1 and $\|\varphi_i\|_{B_2} \leqslant C$ for all $i \in I$ where $\|\cdot\|_{B_2}$ denotes the Herz–Schur norm. The infimum of all C for which such a net exists, is called the *Cowling–Haagerup constant* of G, usually denoted $\Lambda_{\mathrm{cb}}(G)$.

The countable, discrete group G has the *Haagerup property* if there is a net $(\varphi_i)_{i\in I}$ of positive definite functions on G converging pointwise to 1 such that each φ_i vanishes at infinity. An equivalent condition is the existence of a conditionally negative definite function $\psi : G \to \mathbb{R}$ such that ψ is proper, i.e. $\{g \in G \mid |\psi(g)| < n\}$ is finite for each $n \in \mathbb{N}$ (see for instance [5, Theorem 2.1.1]). It follows from Schoenberg's Theorem that given such a ψ, the family $(e^{-t\psi})_{t>0}$ witnesses the Haagerup property.

For a general treatment of weak amenability and the Haagerup property, including examples of groups with and without these properties, we refer the reader to [4].

Since positive definite functions are also Herz–Schur multipliers with norm 1, it is clear that amenability is stronger than both weak amenability with (Cowling–Haagerup) constant 1 and the Haagerup property. A natural question to ask is how weak amenability and the Haagerup property are related. For a long time the known examples of weakly amenable groups with constant 1 also had the Haagerup property and vice versa. Also, the groups that were known to not be weakly amenable also failed the Haagerup property. So it seemed natural to ask if the Haagerup property is equivalent to weak amenability with constant 1. This turned out to be false, and the first counterexample was the wreath product $\mathbb{Z}/2 \wr \mathbb{F}_2$. This group is defined as the semidirect product $(\bigoplus_{\mathbb{F}_2} \mathbb{Z}/2) \rtimes \mathbb{F}_2$, where the action $\mathbb{F}_2 \curvearrowright \bigoplus_{\mathbb{F}_2} \mathbb{Z}/2$ is the shift. In [6] it is shown that the group $\mathbb{Z}/2 \wr \mathbb{F}_2$ has the Haagerup property, and in [12, Corollary 2.12] it was proved that $\mathbb{Z}/2 \wr \mathbb{F}_2$ is not weakly amenable with constant 1. In fact, the group is not even weakly amenable [11, Corollary 4].

It is still an open question if groups that are weakly amenable with constant 1 have the Haagerup property. It may be formulated as follows. Given a net $(\varphi_i)_{i\in I}$ of finitely supported functions on G such that $\|\varphi_i\|_{B_2} \leqslant 1$ and $\varphi_i \to 1$ pointwise, does there exist a proper, conditionally negative definite function on G? We do not answer this question here, but we consider the following related problem. If we replace the condition that each φ_i is finitely supported with the condition that φ_i vanishes at infinity, what can then be said? We make the following definition.

Definition 1.1. A discrete group G has the *weak Haagerup property* if there exist $C > 0$ and a net $(\varphi_i)_{i\in I}$ of Herz–Schur multipliers on G converging pointwise to 1 such that each φ_i vanishes at infinity and satisfies $\|\varphi_i\|_{B_2} \leqslant C$. If we may take $C = 1$, then G has the weak Haagerup property *with constant* 1.

A priori the weak Haagerup property is even less tangible than weak amenability, but the point is that with the weak Haagerup property with constant 1, we can assume that the net in question is a semigroup of the form $(e^{-t\varphi})_{t>0}$, as the following holds.