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Cubic column relations in truncated moment problems

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Abstract

For the truncated moment problem associated to a complex sequence $\gamma^{(2n)} = \{\gamma_{ij}\}_{i,j \in Z_+, i+j \leqslant 2n}$ to have a representing measure μ, it is necessary for the moment matrix $M(n)$ to be positive semidefinite, and for the algebraic variety $\mathcal{V}_\gamma$ to satisfy $\operatorname{rank} M(n) \leqslant \operatorname{card} \mathcal{V}_\gamma$ as well as a consistency condition: the Riesz functional vanishes on every polynomial of degree at most $2n$ that vanishes on $\mathcal{V}_\gamma$. In previous work with L. Fialkow and H.M. Möller, the first named author proved that for the extremal case ($\operatorname{rank} M(n) = \operatorname{card} \mathcal{V}_\gamma$), positivity and consistency are sufficient for the existence of a representing measure. In this paper we solve the truncated moment problem for *cubic* column relations in $M(3)$ of the form $Z^3 = itZ + u\bar{Z}$ ($u, t \in \mathbb{R}$); we do this by checking consistency. For (u, t) in the open cone determined by $0 < |u| < t < 2|u|$, we first prove that the algebraic variety has exactly 7 points and $\operatorname{rank} M(3) = 7$; we then apply the above mentioned result to obtain a concrete, computable, necessary and sufficient condition for the existence of a representing measure.

Keywords: Truncated moment problem; Cubic column relation; Algebraic variety; Riesz functional; Harmonic polynomial

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1. Introduction

Given a collection of complex numbers $\gamma \equiv \gamma^{(2n)}$: $\gamma_{00}, \gamma_{01}, \gamma_{10}, \dots, \gamma_{0,2n}, \gamma_{1,2n-1}, \dots, \gamma_{2n-1,1}, \gamma_{2n,0}$, with $\gamma_{00} > 0$ and $\gamma_{ji} = \bar{\gamma}_{ij}$, the *truncated complex moment problem* (TCMP) consists of finding a positive Borel measure μ supported in the complex plane $\mathbb{C}$ such that $\gamma_{ij} = \int \bar{z}^i z^j \, d\mu$ $(0 \leqslant i + j \leqslant 2n)$; γ is called a *truncated moment sequence* (of order $2n$) and μ is called a *representing measure* for γ. If, in addition, we require $\operatorname{supp} \mu \subseteq K$ (closed) $\subseteq \mathbb{C}$, we speak of the K-TCMP. Naturally associated with each TCMP is a moment matrix $M(n) \equiv M(n)(\gamma)$, whose concrete construction will be given in Subsection 1.1. $M(n)$ detects the positivity of the *Riesz functional* $p \mapsto \sum_{ij} a_{ij} \gamma_{ij}$ $(p(z,\bar{z}) \equiv \sum_{ij} a_{ij} \bar{z}^i z^j)$ on the cone generated by the collection $\{p\bar{p}\colon p \in \mathbb{C}[z,\bar{z}]\}$. In addition to its importance for applications, a complete solution of TCMP would readily lead to a solution of the *full* moment problem, via a weak-* convergence argument, as shown by J. Stochel [39]. While we primarily focus on truncated moment problems, the full moment problem (in one or several variables) has been widely studied; see, for example, [1,2,15, 21–23,25–31,33–37,40–42].

Building on previous work for the case of *real* moments, several years ago the first named author and L. Fialkow introduced in [5–7] an approach to TCMP based on matrix positivity and extension, combined with a new "functional calculus" for the columns of $M(n)$ (labeled $1, Z, \bar{Z}, Z^2, \bar{Z}Z, \bar{Z}^2, \dots$). This allowed them to show that TCMP is soluble in the following cases:

- (i) TCMP is of *flat data* type [5], i.e., $\operatorname{rank} M(n) = \operatorname{rank} M(n-1)$ (this case subsumes all previous results for the Hamburger, Stieltjes, Hausdorff, and Toeplitz truncated moment problems [4]);
- (ii) the column $\bar{Z}$ is a linear combination of the columns 1 and Z [6, Theorem 2.1];
- (iii) for some $k \leqslant [n/2] + 1$, the analytic column Z^k is a linear combination of columns corresponding to monomials of lower degree [6, Theorem 3.1];
- (iv) the analytic columns of $M(n)$ are linearly dependent and span $\mathcal{C}_{M(n)}$, the column space of $M(n)$ [5, Corollary 5.15];
- (v) $M(n)$ is singular and subordinate to conics [9–12];
- (vi) $M(n)$ admits a rank-preserving moment matrix extension $M(n+1)$, i.e., an extension $M(n+1)$ which is flat [13];
- (vii) $M(n)$ is extremal, i.e., $\operatorname{rank} M(n) = \operatorname{card} \mathcal{V}(\gamma^{(2n)})$, where $\mathcal{V}(\gamma) \equiv \mathcal{V}(\gamma^{(2n)})$ is the algebraic variety of γ (see Subsection 1.2 below) [14].

The common feature of the above mentioned cases is the presence, at the level of $\mathcal{C}_{M(n)}$, of algebraic conditions implied by the existence of a representing measure with support in a proper real algebraic subset of the plane. However, the chief attraction of the truncated moment problem (TMP) is its naturalness: since the data set is finite, we can apply "finite" techniques, grounded in finite dimensional operator theory, linear algebra, and algebraic geometry, to develop algorithms for explicitly computing finitely atomic representing measures.

1.1. The truncated complex moment problem

Given a collection of complex numbers $\gamma \equiv \gamma^{(2n)}$: $\gamma_{00}, \gamma_{01}, \gamma_{10}, \dots, \gamma_{0,2n}, \gamma_{1,2n-1}, \dots, \gamma_{2n-1,1}, \gamma_{2n,0}$, with $\gamma_{00} > 0$ and $\gamma_{ji} = \bar{\gamma}_{ij}$, the associated moment matrix $M(n) \equiv M(n)(\gamma)$ is built as follows: