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On the torsion function with Robin or Dirichlet boundary conditions

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Abstract

For $p \in (1, +\infty)$ and $b \in (0, +\infty]$ the p-torsion function with Robin boundary conditions associated to an arbitrary open set $\Omega \subset \mathbb{R}^m$ satisfies formally the equation $-\Delta_p = 1$ in Ω and $|\nabla u|^{p-2}\frac{\partial u}{\partial n} + b|u|^{p-2}u = 0$ on $\partial\Omega$. We obtain bounds of the L^∞ norm of u *only* in terms of the bottom of the spectrum (of the Robin p-Laplacian), b and the dimension of the space in the following two extremal cases: the linear framework (corresponding to $p = 2$) and arbitrary $b > 0$, and the non-linear framework (corresponding to arbitrary $p > 1$) and Dirichlet boundary conditions ($b = +\infty$). In the general case, $p \neq 2$, $p \in (1, +\infty)$ and $b > 0$ our bounds involve also the Lebesgue measure of Ω.

Keywords: Torsion function; Robin boundary conditions; p-Laplacian

1. Introduction

Let Ω be an open set in Euclidean space $\mathbb{R}^m$ with non-empty boundary $\partial\Omega$, and let the torsion function $u : \Omega \to \mathbb{R}$ be the unique weak solution of

$$-\Delta u = 1, \qquad u|_{\partial\Omega} = 0.$$

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If Ω has finite measure the solution is obtained in the usual framework of the Lax–Milgram theorem, while if Ω has infinite measure then $1 \notin H^{-1}(\Omega)$ and u is defined as the supremum over all balls of the torsion functions associated to $\Omega \cap B$.

The torsional rigidity is the set function defined by

$$P(\Omega) = \int_{\Omega} u. \tag{1}$$

Since $u \geqslant 0$ we have that $P(\Omega)$ takes values in the non-negative extended real numbers, and that $P(\Omega) = \|u\|_{L^1(\Omega)}$, whenever u is integrable. Both the torsion function and the torsional rigidity arise in many areas of mathematics, for example in elasticity theory [2,19,23], in heat conduction [5], in the definition of gamma convergence [8,18], in the study of minimal submanifolds [21] etc. The connection with probability theory is as follows. Let $(B(s),\ s \geqslant 0;\ \mathbb{P}_x,\ x \in \mathbb{R}^m)$ be Brownian motion with generator Δ, and let

$$T_\Omega = \inf\{s \geqslant 0:\ B(s) \in \mathbb{R}^m \setminus \Omega\}$$

be the first exit time of Brownian motion from Ω. Then [24]

$$u(x) = \mathbb{E}_x[T_\Omega], \quad x \in \Omega.$$

Let λ be the bottom of the spectrum of the Dirichlet Laplacian acting in $L^2(\Omega)$.

In [6,7] it was shown that $u \in L^\infty(\Omega)$ if and only if $\lambda > 0$. If $\lambda > 0$ then

$$\lambda^{-1} \leqslant \|u\|_{L^\infty(\Omega)} \leqslant (4 + 3m \log 2)\lambda^{-1}. \tag{2}$$

Previous results of this nature were obtained in Theorem 1 of [3] for open, simply connected, planar sets Ω. The question of the sharp constant in the upper bound in the right hand side of (2) for these sets was addressed in [3,4].

In this paper we consider the torsion function u_b for the Laplacian with Robin boundary conditions. The Robin Laplacian is generated by the quadratic form

$$\mathcal{Q}_b(u, v) = \int_{\Omega} \nabla u.\nabla v + b \int_{\partial\Omega} uv\, d\mathcal{H}^{m-1}, \quad u, v \in W^1_{2,2}(\Omega, \partial\Omega),$$

where $\mathcal{H}^{m-1}$ denotes the $(m-1)$-dimensional Hausdorff measure on $\partial\Omega$, and b is a strictly positive constant. This quadratic form defined on $W^1_{2,2}(\Omega, \partial\Omega)$ is closed. The unique self-adjoint operator generated by $\mathcal{Q}_b$ is the Robin Laplacian which formally satisfies the boundary condition

$$\frac{\partial u}{\partial n} + bu = 0, \quad x \in \partial\Omega, \tag{3}$$

where n denotes the outward unit normal, and $\frac{\partial}{\partial n}$ is the normal derivative. The torsion function u_b is the unique weak solution of $-\Delta u = 1$ with boundary condition (3). For convenience we put $q_b(u) = \mathcal{Q}_b(u, u)$. It is well known that $W^1_{2,2}(\Omega) = W^{1,2}(\Omega)$ if Ω is bounded and $\partial\Omega$ is Lipschitz. See [22] for details. However, as all our results are for arbitrary open sets in $\mathbb{R}^m$ we