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Uniformly bounded representations and exact groups

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Abstract

We characterize groups with Guoliang Yu's property A (i.e., exact groups) by the existence of a family of uniformly bounded representations which approximate the trivial representation.

Keywords: Property A; Uniformly bounded representation; Exact groups; Fell topology

Property A is a large scale geometric property that can be viewed as a weak counterpart of amenability. It was shown in [12], that for a finitely generated group property A implies the Novikov conjecture. It was also quickly realized that this notion has many other applications and interesting connections, see [10,9].

A well-known characterization of amenability states that the constant function 1 on G, as a coefficient of the trivial representation, can be approximated by diagonal, finitely supported coefficients of the left regular representation of G on $\ell_2(G)$. In this note we prove a counterpart of this result for groups with property A in terms of uniformly bounded representations. A representation π of a group G on a Hilbert space H is said to be uniformly bounded if $\sup_{g\in G} \|\pi_g\|_{B(H)} < \infty$.

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Theorem 1. *Let G be a finitely generated group equipped with a word length function. G has property A (i.e., G is exact) if and only if for every $\varepsilon > 0$ there exists a uniformly bounded representation π of G on a Hilbert space H, a vector $v \in H$ and a constant $S > 0$ such that*

(1) $\|\pi_g v\| = 1$ *for all* $g \in G$,
(2) $|1 - \langle \pi_g v, \pi_h v\rangle| \leqslant \varepsilon$ *if* $|g^{-1}h| \leqslant 1$,
(3) $\langle \pi_g v, \pi_h v\rangle = 0$ *if* $|g^{-1}h| \geqslant S$.

Alternatively, the second condition can be replaced by an almost-invariance condition: $\|\pi_g v - \pi_h v\| \leqslant \varepsilon$ if $|g^{-1}h| \leqslant 1$. Another characterization of property A in this spirit, involving convergence for isometric representations on Hilbert C^*-modules was studied in [4].

Recall that the Fell topology on the unitary dual is defined using convergence of coefficients of unitary representations. Theorem 1 states that the trivial representation can be approximated by uniformly bounded representations, in a fashion similar to Fell's topology.

Similar phenomena were considered by M. Cowling [2,3] in the case of the Lie group $\mathrm{Sp}(n, 1)$. Recall that $\mathrm{Sp}(n, 1)$ has property (T), and thus the trivial representation is an isolated point among the equivalence classes of unitary representations in the Fell topology. Cowling showed that nevertheless, for $\mathrm{Sp}(n, 1)$ the trivial representation can be approximated by uniformly bounded representations in a certain sense. Theorem 1 gives a similar statement for all discrete groups with property A. Recall that almost all known groups with property (T) are known to have property A. In particular, the groups $\mathrm{SL}_n(\mathbb{Z})$, $n \geqslant 3$, satisfy property A [6].

Moreover, under a stronger assumption that the group has Hilbert space compression strictly greater than $1/2$ in the sense of [5], we obtain a path of uniformly bounded representations, whose coefficients continuously interpolate between the trivial and the left regular representations.

Theorem 1 suggests the possibility of negating property A using strengthened forms of Kazhdan's property that applies to uniformly bounded representations.

Question 1. Are there finitely generated groups satisfying a sufficiently strong version of property (T) for uniformly bounded representations, so that these groups cannot have property A?

Certain versions of such a property (T) for uniformly bounded representations were considered by Cowling [2,3], but they would not apply directly in our case. Construction of new examples of finitely generated groups without property A is a major open problem in coarse geometry, with possible applications in operator algebras, index theory and topology of manifolds.

1. Uniformly bounded representations and property A

Let H_0 be a Hilbert space with scalar product $\langle\cdot,\cdot\rangle_0$, and let T be a bounded, positive, self-adjoint operator on H_0. We additionally assume that T has a spectral gap; that is, there exists $\lambda > 0$ such that

$$\langle v, Tv\rangle_0 \geqslant \lambda \langle v, v\rangle_0 \tag{1}$$

for every $v \in H_0$.