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How to recognize convexity of a set from its marginals

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Abstract

We investigate the regularity of the marginals onto hyperplanes for sets of finite perimeter. We prove, in particular, that if a set of finite perimeter has log-concave marginals onto a.e. hyperplane then the set is convex. Our proof relies on measuring the perimeter of a set through a Hilbertian fractional Sobolev norm, a fact that we believe has its own interest.
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1. Introduction

Let $\varphi : \mathbb{R}^n \to [0, +\infty)$ be a log-concave function, that is, φ is of the form e^{-V} for some convex function $V : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$. A well-known consequence of the Prékopa–Leindler inequality states that the marginals of φ onto any hyperplane are log-concave (see for instance [6, Sections 10–11]): more precisely, for any direction $\mathbf{e} \in \mathbb{S}^{n-1}$ let $\pi_{\mathbf{e}} : \mathbb{R}^n \to \mathbf{e}^\perp$ denote the orthogonal projection onto the hyperplane $\mathbf{e}^\perp := \{x \in \mathbb{R}^n\colon\ \mathbf{e} \cdot x = 0\}$, and define

$$\varphi_{\mathbf{e}} : \mathbf{e}^\perp \to \mathbb{R}, \qquad \varphi_{\mathbf{e}}(x) := \int_{\mathbb{R}} \varphi(x + t\mathbf{e})\, dt.$$

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Then $\varphi_{\mathbf{e}}$ is log-concave, i.e., $\varphi_{\mathbf{e}} = e^{-W_{\mathbf{e}}}$ for some convex function $W_{\mathbf{e}} : \mathbf{e}^\perp \simeq \mathbb{R}^{n-1} \to \mathbb{R} \cup \{+\infty\}$. In particular, if E is a convex set and we denote by $\mathbf{1}_E$ the characteristic function of E (that is $\mathbf{1}_E(x) = 1$ if $x \in E$, $\mathbf{1}_E(x) = 0$ if $x \notin E$), then $\mathbf{1}_E$ is log-concave, which implies that, for any $\mathbf{e} \in \mathbb{S}^{n-1}$,

$$w_{\mathbf{e}} : \mathbf{e}^\perp \to \mathbb{R}, \qquad w_{\mathbf{e}}(x) := \int_{\mathbb{R}} \mathbf{1}_E(x + t\mathbf{e})\, dt \tag{1.1}$$

is log-concave. Actually, by the Brunn–Minkowski inequality, an even stronger result is true, namely $w_{\mathbf{e}}$ is concave. (We refer to [6] for more details.)

In [5], Falconer proved that a converse of the previous statements is true: if a compact set E has concave marginals onto a.e. hyperplane $\mathbf{e}^\perp$, then it is convex. The aim of this paper is to show that (under rather weak regularity assumptions on E) this converse statement is still true under weaker assumptions on the marginals: namely, we prove that if the marginals have convex support and are uniformly Lipschitz strictly inside their support, then the set is convex. In particular, this implies that if a set has log-concave marginals onto a.e. hyperplane $\mathbf{e}^\perp$ then it is convex. As we will also discuss below, this fact is false if we do not restrict to characteristic functions of sets: it is possible to construct examples of functions whose marginals are log-concave (actually, even concave) but the functions themselves are not.

To state our result, let us introduce some notation. Given a Borel set E, let $w_{\mathbf{e}}$ be as in (1.1), and define the set $A_{\mathbf{e}} := \{w_{\mathbf{e}} > 0\} \subset \mathbf{e}^\perp$. (Notice that if $w_{\mathbf{e}}$ is log-concave then $A_{\mathbf{e}}$ is convex.) Also, for any $\delta > 0$ we set $A_{\mathbf{e}}^\delta := \{x \in A_{\mathbf{e}}\colon \operatorname{dist}(x, \partial A_{\mathbf{e}}) \geqslant \delta\}$. We recall that a set E is of *finite perimeter* if the distributional derivative $\nabla \mathbf{1}_E$ of $\mathbf{1}_E$ is a finite measure, that is

$$\int_{\mathbb{R}^n} |\nabla \mathbf{1}_E| < \infty.$$

Also, we use $\mathcal{H}^k$ to denote the k-dimensional Hausdorff measure. Here is our main result:

Theorem 1.1. *Let $E \subset \mathbb{R}^n$ be a bounded set of finite perimeter and assume that $A_{\mathbf{e}}$ is convex for $\mathcal{H}^{n-1}$-a.e. $\mathbf{e} \in \mathbb{S}^{n-1}$. Suppose further that $w_{\mathbf{e}}$ is locally Lipschitz inside $A_{\mathbf{e}}$ for $\mathcal{H}^{n-1}$-a.e. $\mathbf{e} \in \mathbb{S}^{n-1}$ and the following uniform bound holds: for any $\delta > 0$ there exists a constant C_δ such that*

$$|\nabla w_{\mathbf{e}}| \leqslant C_\delta \text{ a.e. inside } A_{\mathbf{e}}^\delta, \quad \text{for } \mathcal{H}^{n-1}\text{-a.e. } \mathbf{e} \in \mathbb{S}^{n-1}.$$

Then E is convex (up to a set of measure zero).

Notice that, since log-concave functions are Lipschitz in the interior of their support, our assumption is weaker than asking that $w_{\mathbf{e}}$ is log-concave for $\mathcal{H}^{n-1}$-a.e. $\mathbf{e} \in \mathbb{S}^{n-1}$. Hence our theorem implies the following[1]:

[1] To be precise, since the bound on the Lipschitz constant of a convex function depends on its L^∞ norm in a slightly larger domain, the bound on the Lipschitz constant for a log-concave function depends on a lower bound on the function itself in a slightly larger domain. Although in general there is no universal bound for a general class of log-concave functions, in our case all the functions $w_{\mathbf{e}}$ arise as marginals of a bounded set, which implies that $A_{\mathbf{e}_k} \to A_{\mathbf{e}}$ whenever

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