Available online at www.sciencedirect.com

ScienceDirect

Journal of Functional Analysis 266 (2014) 1702–1715

JOURNAL OF Functional Analysis

www.elsevier.com/locate/jfa

Closed subgroups as Ditkin sets

Antoine Derighetti

Received 2 May 2013; accepted 6 November 2013

Available online 28 November 2013

Communicated by D. Voiculescu

Abstract

We prove that every closed subgroup of a locally compact group is locally p-Ditkin for $1 < p < \infty$.

Keywords: Abstract harmonic analysis; Locally compact groups; Operator theory

1. Introduction

Let G be a locally compact group and H a closed subgroup. The following result is well known for locally compact abelian group: whenever $f \in L^1(\widehat{G})$ (here $\widehat{G}$ denotes the Pontrjagin dual of G) is such that $\widehat{f}$ vanishes on H and $u \in L^\infty(\widehat{G})$ is such that $sp(f * u) \subset H$, then $f * u = 0$. The aim of this work is to generalize this result to arbitrary non-commutative locally compact groups and arbitrary closed subgroups. We replace $L^1(\widehat{G})$ by the Figà-Talamanca–Herz algebra $A_p(G)$ and $L^\infty(\widehat{G})$ by the Banach algebra $CV_p(G)$ of all convolution operators of $L^p(G)$ where $1 < p < \infty$. In the classical case $p = 2$ our statement (Corollary 6) is the following: whenever $u \in A_p(G)$ is such that u vanishes on H and $T \in CV_p(G)$ is such that the support of uT is contained in H, then $uT = 0$. Equivalently if $u \in A_p(G)$ vanishes on H and $Y \in CV_p(G)$ has support in H then the equation $uX = Y$ has no solution $X \in CV_p(G)$ unless $Y = 0$.

We proved this for closed normal subgroups in [4], and for neutral subgroups in collaboration with J. Delaporte in [2]. For G amenable and H arbitrary closed subgroup, the result is due to B. Forrest, E. Kaniuth, A.T. Lau and N. Spronk [7]. J. Ludwig and L. Turowska proved [9] for G, a second countable locally compact group, that every closed subgroup is locally 2-Ditkin. In [5] we obtained that every closed amenable subgroup of a locally compact group is locally p-Ditkin.

E-mail address: antoine.derighetti@epfl.ch.

0022-1236/$ – see front matter
http://dx.doi.org/10.1016/j.jfa.2013.11.001

2. Locally p-Ditkin sets

For $f \in \mathbb{C}^G$, a and $x \in G$ we put $\check{f}(x) = f(x^{-1})$, ${}_a f(x) = f(ax)$ and $f_a(x) = f(xa)$. If A is a subset of G $Res_A\, f$ denotes the map of A into $\mathbb{C}$ $a \mapsto f(a)$. We denote by m_G a left invariant Haar measure and by Δ_G the modular function of G. We write $\mathcal{L}^1(G)$ for the $\mathbb{C}$-vector space of all $\varphi \in \mathbb{C}^G$ which are integrable. If $1 < p < \infty$ $\mathcal{L}^p(G)$ is the vector space of all $\varphi \in \mathbb{C}^G$ such that φ is measurable and $|\varphi|^p$ is integrable. For $1 \leqslant p < \infty$ and $f \in \mathcal{L}^p(G)$ we put $N_p(f) = (\int |f(x)|^p\, dx)^{1/p}$. For $f \in \mathbb{C}^G$ we denote by $[f]$ the set of all $g \in \mathbb{C}^G$ with $g(x) = f(x)$ almost everywhere. For $f \in \mathcal{L}^p(G)$ we set $\|[f]\|_p = N_p(f)$ and $L^p(G) = \{[f] \mid f \in \mathcal{L}^p(G)\}$. For $1 < p < \infty$, $p' = p/(p-1)$, $f \in \mathcal{L}^p(G)$ and $g \in \mathcal{L}^{p'}(G)$ we put $\langle [f], [g] \rangle = \int f(x)\overline{g(x)}\, dx$.

Let $1 < p < \infty$. We denote by $\mathcal{A}_p(G)$ the set of all pairs $((r_n), (s_n))$ where (r_n) is a sequence of $\mathcal{L}^p(G)$ and (s_n) is a sequence of $\mathcal{L}^{p'}(G)$ such that $\sum N_p(r_n) N_{p'}(s_n)$ converges. By definition $A_p(G)$ is the set of all $u \in \mathbb{C}^G$ such that there is $((r_n), (s_n)) \in \mathcal{A}_p(G)$ with $u(x) = \sum (\overline{r_n} * \check{s}_n)(x)$ for every $x \in G$. We put

$$\|u\|_{A_p(G)} = \inf\left\{ \sum N_p(r_n) N_{p'}(s_n) \;\middle|\; \big((r_n), (s_n)\big) \in \mathcal{A}_p(G),\ u = \sum \overline{r_n} * \check{s}_n \right\}.$$

We refer to [6] for complementary results on $A_p(G)$.

Definition 1. Let G be a locally compact group, $1 < p < \infty$ and F a closed subset of G. We say that F is a locally p-Ditkin subset of G if for every $u \in A_p(G) \cap C_{00}(G)$ vanishing on F and for every $\varepsilon > 0$ there is $v \in A_p(G) \cap C_{00}(G)$ with $\operatorname{supp} v \cap F = \emptyset$ and $\|u - uv\| < \varepsilon$.

In [4, p. 102] we proved that a closed subset F of G is locally p-Ditkin if and only if for every $T \in CV_p(G)$ and every $u \in A_p(G)$ with $\operatorname{supp} uT \subset F$ and $Res_F\, u = 0$ we have $uT = 0$.

3. The action of $A_p(G)$ on $\mathcal{L}(L^p(G))$

Denote by $\mathcal{T}$ the trace class operators of $L^p(G)$ and by $\mathcal{L}$ the bounded operators of $L^p(G)$. For $((r_n), (s_n)) \in \mathcal{A}_p(G)$ we denote by $T_{((r_n),(s_n))}$ the trace class operator $\langle T_{((r_n),(s_n))} f, g \rangle = \sum \langle [r_n], g \rangle \langle f, [\overline{s_n}] \rangle$. Putting

$$F_{((r_n),(s_n))}(x, y) = \sum r_n(x) s_n(y)$$

if $\sum r_n(x) s_n(y)$ converges and 0 otherwise, we get

$$\big\langle T_{((r_n),(s_n))}[f], [g] \big\rangle = \int\limits_{G \times G} F_{((r_n),(s_n))}(x, y) \overline{g(x)} f(y)\, dx\, dy.$$

This integral formula permits to associate in a bilinear way to $\varphi \in C^b(G \times G)$ and $S \in \mathcal{T}$ an operator φS of $\mathcal{L}$ with $\|\varphi S\| \leqslant \|\varphi\|_\infty \|S\|_{\mathcal{T}}$. Setting for $\psi : G \times G \to \mathbb{C}$ $(\varXi \psi)(x, y) = \psi(y, x)$ and for $\varphi : G \to \mathbb{C}$ $(M_G \varphi)(x, y) = \varphi(y x^{-1})$, we get for $u \in A_p(G)$ and $S \in \mathcal{T}$ $\varXi M_G u S \in \mathcal{T}$ and $\|\varXi M_G u S\|_{\mathcal{T}} \leqslant \|u\|_{A_p(G)} \|S\|_{\mathcal{T}}$. The pairing of $\mathcal{L}$ with $\mathcal{T}$ is defined in the following way: for $U \in \mathcal{L}$ and $S \in \mathcal{T}$ we put