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Criteria of spectral gap for Markov operators ☆

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Abstract

Let $(E, \mathcal{F}, \mu)$ be a probability space, and let P be a Markov operator on $L^2(\mu)$ with 1 a simple eigenvalue such that $\mu P = \mu$ (i.e. μ is an invariant probability measure of P). Then $\hat{P} := \frac{1}{2}(P + P^*)$ has a spectral gap, i.e. 1 is isolated in the spectrum of $\hat{P}$, if and only if

$$\|P\|_\tau := \lim_{R\to\infty} \sup_{\mu(f^2)\leqslant 1} \mu\big(f(Pf - R)^+\big) < 1.$$

This strengthens a conjecture of Simon and Høegh-Krohn on the spectral gap for hyperbounded operators solved recently by L. Miclo in [10]. Consequently, for a symmetric, conservative, irreducible Dirichlet form on $L^2(\mu)$, a Poincaré/log-Sobolev type inequality holds if and only if so does the corresponding defective inequality. Extensions to sub-Markov operators and non-conservative Dirichlet forms are also presented.
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1. Introduction

Let $(E, \mathcal{F}, \mu)$ be a probability space. Let P be a Markov operator on $L^2(\mu)$ (i.e. P is a linear operator on $L^2(\mu)$ such that $P1 = 1$ and $f \geqslant 0$ implies $Pf \geqslant 0$) such that $\mu P = \mu$ (i.e. μ is an invariant probability measure of P). Let $\hat{P} = \frac{1}{2}(P + P^*)$ be the additive symmetrization of P,

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where P^* is the adjoint operator of P on $L^2(\mu)$. Assuming that 1 is a simple eigenvalue of P, we aim to investigate the existence of spectral gap of $\hat{P}$ (i.e. 1 is an isolated point in $\sigma(\hat{P})$, the spectrum of $\hat{P}$).

It is well known by the ergodic theorem that 1 is a simple eigenvalue of P if and only if P is ergodic, i.e. for $f \in L^2(\mu)$,

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} P^k f = \mu(f) \quad \text{holds in } L^2(\mu).$$

Moreover, the ergodicity (i.e. 1 is a simple eigenvalue) is also equivalent to the μ-essential irreducibility (or resolvent-positive-improving property, see [20]) of P:

$$\sum_{n=1}^{\infty} \mu\big(1_A P^n 1_B\big) > 0, \quad \mu(A), \mu(B) > 0. \tag{1.1}$$

Indeed, if for some $f \in L^2(\mu)$ with $\mu(f) = 0$ and $\mu(f^2) = 1$ such that $Pf = f$, then by the Jensen inequality we have $Pf^+ \geqslant (Pf)^+ = f^+$. But $\mu(Pf^+) = \mu(f^+)$, we conclude that $Pf^+ = f^+$. Then for any $\varepsilon > 0$ such that $\mu(f < -\varepsilon), \mu(f > \varepsilon) > 0$, we have

$$\mu\big(1_{\{f<-\varepsilon\}} P^n 1_{\{f>\varepsilon\}}\big) \leqslant \frac{1}{\varepsilon^2}\mu\big(f^- P^n f^+\big) = \frac{1}{\varepsilon^2}\mu\big(f^- f^+\big) = 0, \quad n \geqslant 1,$$

so that (1.1) does not hold. On the other hand, if there exist $A, B \in \mathcal{F}$ with $\mu(A), \mu(B) > 0$ such that $\mu(1_B P^n 1_A) = 0$ for all $n \geqslant 1$, then the class

$$\mathcal{C} := \big\{0 \leqslant f \leqslant 1\colon \mu\big(1_B P^n f\big) = 0,\ n \geqslant 1\big\}$$

contains the non-trivial function 1_A. Since the family is bounded in $L^2(\mu)$, we may take a sequence $\{f_n\} \subset \mathcal{C}$ which converges weakly to some $f \in \mathcal{C}$ such that $\mu(f) = \sup_{g\in\mathcal{C}} \mu(g)$. As $f \vee (Pf)$ is also in $\mathcal{C}$, we have $\mu((Pf) \vee f) = \mu(f) = \mu(Pf)$. Thus, $Pf = f$. Since $\mu(f) \geqslant \mu(A) > 0$ and $\mu(1_B Pf) = 0$, we conclude that $f - \mu(f)$ is a non-trivial eigenfunction of P with respect to 1, so that 1 is not a simple eigenvalue of P.

When P is symmetric, the spectrum $\sigma(P)$ of P is contained in $[-1, 1]$. Then P has a spectral gap if P is ergodic and $\sigma(P) \subset \{1\} \cup [-1, \theta]$ for some $\theta \in [-1, 1)$; or equivalently, the Poincaré inequality

$$\mu\big(f^2\big) \leqslant \frac{1}{1-\theta}\mu\big(f(f - Pf)\big) + \mu(f)^2, \quad f \in L^2(\mu) \tag{1.2}$$

holds. When P is non-symmetric, (1.2) is equivalent to $\sigma(\hat{P}|_{\mathcal{H}_0}) \subset [-1, \theta]$, where $\mathcal{H}_0 := \{f \in L^2(\mu)\colon \mu(f) = 0\}$. Thus, (1.2) holds for some $\theta \in [-1, 1)$ if and only if $\hat{P}$ has a spectral gap.

Recall that P is called hyperbounded if for some $p > 2$

$$\|P\|_{2\to p} := \sup_{\mu(f^2)\leqslant 1} \mu\big(|Pf|^p\big)^{\frac{1}{p}} < \infty.$$

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