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Reflection positivity and conformal symmetry

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Abstract

A reflection positive Hilbert space is a triple $(\mathcal{E}, \mathcal{E}_+, \theta)$, where $\mathcal{E}$ is a Hilbert space, θ a unitary involution and $\mathcal{E}_+$ a closed subspace on which the hermitian form $\langle v, w\rangle_\theta := \langle \theta v, w\rangle$ is positive semidefinite. For a triple (G, τ, S), where G is a Lie group, τ an involutive automorphism of G and S a subsemigroup invariant under the involution $s \mapsto s^\sharp = \tau(s)^{-1}$, a unitary representation π of G on $(\mathcal{E}, \mathcal{E}_+, \theta)$ is called reflection positive if $\theta\pi(g)\theta = \pi(\tau(g))$ and $\pi(S)\mathcal{E}_+ \subseteq \mathcal{E}_+$. This is the first in a series of papers in which we develop a new and systematic approach to reflection positive representations based on reflection positive distributions and reflection positive distribution vectors. This approach is most natural to obtain classification results, in particular in the abelian case. Among the tools we develop is a generalization of the Bochner–Schwartz Theorem to positive definite distributions on open convex cones. We further illustrate our techniques with a non-abelian example by constructing reflection positive distribution vectors for complementary series representations of the conformal group $\mathrm{O}^+_{1,n+1}(\mathbb{R})$ of the sphere $\mathbb{S}^n$.

Keywords: Lie group; Unitary representation; Reflection positivity; Conformal group

Contents

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0. Introduction

The concept of reflection positivity has its origins in the work of Osterwalder and Schrader [45,46] on constructive quantum field theory and duality between unitary representations of the Euclidean motion group $E_n = \mathrm{O}_n(\mathbb{R}) \ltimes \mathbb{R}^n$ and the Poincaré group $P_n = \mathrm{O}^+_{1,n-1}(\mathbb{R}) \ltimes \mathbb{R}^{1,n-1}$ of affine isometries of n-dimensional Minkowski space. Here $\mathrm{O}^+_{1,n-1}(\mathbb{R})$ is the group preserving the Lorentz form $(t,x) \mapsto t^2 - \|x\|^2$ and mapping the forward light cone

$$\Omega = \{(t,x) \mid t^2 - \|x\|^2 > 0,\ t > 0\}$$

onto itself. Multiplying the time coordinate t by $i = \sqrt{-1}$ transforms the Lorentz form into $-t^2 - \|x\|^2 = -\|(t,x)\|^2$ and this sets up a duality between the groups P_n and E_n.

On the mathematical side this duality can be made precise as follows. If $\mathfrak{g}$ is a Lie algebra with an involutive automorphism τ, then we have the τ-eigenspace decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q} = \ker(\tau - \mathbf{1}) \oplus \ker(\tau + \mathbf{1})$ and the subspace $\mathfrak{g}^c := \mathfrak{h} \oplus i\mathfrak{q}$ of the complexification $\mathfrak{g}_{\mathbb{C}}$ of $\mathfrak{g}$ is another real form. We thus obtain a duality between the pairs $(\mathfrak{g}, \tau)$ and $(\mathfrak{g}^c, \tau)$. At the core of the notion of reflection positivity is the idea that this duality can sometimes be implemented on the level of unitary representations. This is quite simple on the Lie algebra level: Let $\mathcal{E}^0$ be a pre-Hilbert space and π be a representation of $\mathfrak{g}$ on $\mathcal{E}^0$ by skew-symmetric operator. We also assume that there exists a unitary operator θ on $\mathcal{E}^0$ with $\theta\pi(x)\theta = \pi(\tau x)$ for $x \in \mathfrak{g}$, and a $\mathfrak{g}$-invariant