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Fourth Moment Theorems for Markov diffusion generators

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Abstract

Inspired by the insightful article [4], we revisit the Nualart–Peccati criterion [13] (now known as the Fourth Moment Theorem) from the point of view of spectral theory of general Markov diffusion generators. We are not only able to drastically simplify all of its previous proofs, but also to provide new settings of diffusive generators (Laguerre, Jacobi) where such a criterion holds. Convergence towards Gamma and Beta distributions under moment conditions is also discussed.

Keywords: Markov operator; Diffusion generator; Eigenfunction; Chaos; Fourth Moment Theorem

Contents

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1. Introduction

In 2005, Nualart and Peccati [13] discovered the surprising fact that any sequence of random variables $\{X_n\}_{n\geqslant 1}$ in a Gaussian chaos of fixed order converges in distribution towards a standard Gaussian random variable if and only if $\mathbb{E}(X_n^2) \to 1$ and $\mathbb{E}(X_n^4) \to 3$. In fact, this result contains the two following important informations of a different nature:

(i) For all non-zero X in a Wiener chaos with order $\geqslant 2$, $\mathbb{E}(X^4) > 3\mathbb{E}(X^2)^2$.
(ii) $\mathbb{E}(X^4) - 3\mathbb{E}(X^2)^2 \approx 0$ if and only if $X \overset{Law}{\approx} \mathcal{N}(0, \mathbb{E}(X^2))$.

This striking discovery, now known as the Fourth Moment Theorem, has been the starting point of a fruitful line of research of which we shall give a quick overview. The proof of the above result given in [13] used the Dambis–Dubins–Schwarz Theorem (see e.g. [16, ch. 5]) and did not provide any estimates. In [12], this phenomenon was translated in terms of Malliavin operators, whereas in [8] these operators were combined with the bounds arising from Stein's method, thus yielding both a short proof and precise estimates in the total variation distance (see also [10]). The main difficulty of the proof consists of establishing the powerful inequality

$$\mathrm{Var}\big(\Gamma(X)\big) \leqslant C\big(\mathbb{E}\big(X^4\big) - 3\mathbb{E}\big(X^2\big)^2\big), \tag{1}$$

where Γ is the carré du champ operator associated with the generator of the Ornstein–Uhlenbeck semigroup, from which one can almost immediately deduce convergence in law towards a standard Gaussian distribution $\mathcal{N}(0, 1)$ (for instance by Stein's lemma). Other proofs of the Fourth Moment Theorem, not necessarily relying on the inequality (1), can be found in [6,11,4]. We also mention [3] for extensions to the free probability setting, [15] for the multivariate setting, [7] for Gamma approximation, as well as [14] for the discrete setting. It is important to note that virtually all the proofs (with the remarkable exception of [4]), make crucial use of the product formula for multiple integrals and thus rely on a very rigid structure of the underlying probability space. As a matter of fact, this approach does not cover other important structures like Laguerre and Jacobi, which are investigated in the present article.

In the recent article [4], M. Ledoux gave another proof of the Fourth Moment Theorem in the general framework of diffusive Markov generators, adopting a purely spectral point of view. In particular, he completely avoids the use of product formulae. Unlike the Wiener space setting, it turns out that in this more general framework it is not sufficient anymore that a random variable X is only an eigenfunction of the diffusion generator for an equality of the type (1) to hold. By imposing additional assumptions, one is thus naturally led to a general definition of chaos.