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Geometric structure of dimension functions of certain continuous fields

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Abstract

For a continuous field of C^*-algebras A, we give a criterion to ensure that the stable rank of A is one. In the particular case of a trivial field this leads to a characterization of stable rank one, completing accomplishments by Nagisa, Osaka and Phillips. Further, for certain continuous fields of C^*-algebras, we study when the Cuntz semigroup satisfies the Riesz interpolation property, and we also analyze the structure of its functionals. As an application, we obtain a positive answer to a conjecture posed by Blackadar and Handelman in a variety of situations.

Keywords: C^*-algebras; Continuous fields; Cuntz semigroup; Dimension functions

0. Introduction

In this paper we analyze structural aspects of certain continuous fields of C^*-algebras. Firstly, in connection with non-commutative dimension, we study conditions that determine the stable rank of such fields to be one. Building on work of [27], we settle this in great generality: The algebra $C(X, D)$ of continuous functions from a one-dimensional compact metric space into a C^*-algebra D has stable rank one if and only if the stable rank of D is one and every hereditary subalgebra B of D has trivial K_1. For general continuous fields of C^*-algebras A the condition requiring each fiber to have no K_1-obstructions (in the above sense) is still a sufficient condition

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for $\mathrm{sr}(A) = 1$, but it is not a necessary condition in the general case, as we show by means of a counterexample.

Secondly, we study the Cuntz semigroup $\mathrm{W}(A)$, an invariant built out of the positive elements in matrix algebras. Recent years of research aiming at the classification of C*-algebras have demonstrated this semigroup to be an important ingredient. Performing Grothendieck's construction on $\mathrm{W}(A)$ gives an ordered abelian group $\mathrm{K}_0^*(A)$, whose structure has been explored in a number of instances. It was shown in [30] that $\mathrm{K}_0^*(A)$ has Riesz interpolation for any C*-algebra A with real rank zero and stable rank one and this, in turn, used the result by Zhang that $\mathrm{K}_0(A)$ has Riesz interpolation whenever A has real rank zero [39]. This naturally leads to the question of determining the largest class $\mathcal{B}$ of algebras such that $\mathrm{K}_0^*(A)$ has Riesz interpolation for all $A \in \mathcal{B}$. It was shown in [10] (see also [38]), that $\mathcal{B}$ contains the class of all unital, simple, separable, exact and $\mathcal{Z}$-stable C*-algebras. Our results show that $\mathcal{B}$ also contains certain continuous fields of C*-algebras.

Since it is known that $\mathrm{K}_0^*(A)$ has interpolation if $\mathrm{W}(A)$ does, it is possible to approach the above question by concentrating on proving the latter. We show that $\mathrm{W}(A)$ has Riesz interpolation if and only if its stabilized version $\mathrm{Cu}(A) = \mathrm{W}(A \otimes \mathbb{K})$ (see [11]) has interpolation, provided that $\mathrm{W}(A)$ lies naturally as a hereditary subsemigroup in $\mathrm{Cu}(A)$, which is known to happen if A has stable rank one [1]. (There are currently no known examples of C*-algebras A such that $\mathrm{W}(A)$ is not hereditary in $\mathrm{Cu}(A)$, see [7].) Taking advantage of our results on the stable rank and of recent advances in the computation of the Cuntz semigroup, we prove for certain continuous fields of C*-algebras A that $\mathrm{K}_0^*(A)$ has interpolation.

Cuntz [12] initiated the study of the functionals on $\mathrm{K}_0^*(A)$, i.e., the normalized, order-preserving group homomorphisms into the reals. These functionals are referred to as dimension functions of A. Blackadar and Handelman posed two conjectures on the geometry of the set of dimension functions on a given C*-algebra A in their 1982 paper [6]. Firstly, they conjectured that the set of dimension functions forms a simplex. Secondly, they conjectured that the set of lower semicontinuous dimension functions is dense in the set of all dimension functions. The relevance of the latter conjecture lies in the fact that the set of lower semicontinuous dimension functions, being in correspondence with the quasitraces in A, is more tractable than the set of all dimension functions.

The second named conjecture was proved in [6] for commutative C*-algebras, while the first conjecture was left completely unanswered. By a result in [19], the set of dimension functions forms a Choquet simplex provided $\mathrm{K}_0^*(A)$ has Riesz interpolation, and this is what was used in [30] and [10] to confirm the first conjecture for the classes of algebras therein. (Further, this conjecture was strengthened in [10] to asking the set of dimension functions to form a Choquet simplex.) Likewise, our results on $\mathrm{K}_0^*(A)$ imply an affirmative answer to the first conjecture for certain classes of continuous fields. The second conjecture was shown to hold for all unital, simple, separable, exact and $\mathcal{Z}$-stable C*-algebras in [10], by giving a suitable representation of their Cuntz semigroup. Similarly, our approach to the second conjecture for continuous fields A is based on representing $\mathrm{K}_0^*(A)$ sufficiently well into the group of affine and bounded functions on the trace space of A.

The paper is organized as follows. Section 1 contains our results on the stable rank of continuous fields, which can be read independently of the rest of the paper. It follows a short section on hereditariness of $\mathrm{W}(A)$ in $\mathrm{Cu}(A)$ for continuous fields of C*-algebras A. Interpolation results are proved in Section 3, before we apply our results in Section 4 to answer the Blackadar–Handelman conjectures affirmatively for the C*-algebras under consideration.