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Journal of Functional Analysis 266 (2014) 2436–2485

JOURNAL OF Functional Analysis

www.elsevier.com/locate/jfa

Boundedness and growth for the massive wave equation on asymptotically anti-de Sitter black holes

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Received 10 June 2013; accepted 13 October 2013

Available online 31 October 2013

Communicated by I. Rodnianski

Abstract

We study the global dynamics of free massive scalar fields on general, globally stationary, asymptotically AdS black hole backgrounds with Dirichlet, Neumann or Robin boundary conditions imposed on ψ at infinity. This class includes the regular Kerr–AdS black holes satisfying the Hawking–Reall bound $r_+^2 > |a|l$. We establish a suitable criterion for linear stability (in the sense of uniform boundedness) of ψ and demonstrate how the issue of stability can depend on the boundary condition prescribed. In particular, in the slowly rotating Kerr–AdS case, we obtain the existence of linear scalar hair (i.e. non-trivial stationary solutions) for suitably chosen Robin boundary conditions.

Keywords: Anti-de Sitter black holes; Klein–Gordon equation

Contents

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0022-1236/$ – see front matter © 2013 Elsevier Inc. All rights reserved.
http://dx.doi.org/10.1016/j.jfa.2013.10.019

1. Introduction

The classical stability properties of asymptotically anti-de Sitter (aAdS) spacetimes have attracted recent attention in the general relativity community. To a large extent, this interest derives from the range of potential instability mechanisms which can be inferred – so far only at the heuristic and numerical level [1,7,4,10] – from the geometry of these spacetimes, in particular from their asymptotic structure. These phenomena are entirely absent in the asymptotically flat case and have culminated in the conjecture that all asymptotically AdS spacetimes (including Kerr–AdS and pure AdS) may be unstable [21]. See [9] for some recent work where this conjecture is being investigated.

From a classical perspective, the crucial feature of aAdS spacetimes is their failure of global hyperbolicity: Despite the fact that null-infinity is "infinitely far away" in the sense that the affine length of null geodesics approaching infinity is indeed infinite, the causal structure of the spacetime also has the following property: Given a spacelike slice Σ, there exist points, p, in $I^+(\Sigma)$ and (complete) past directed causal curves from p, which do not intersect Σ. This suggests that hyperbolic equations on such manifolds will, in general, require boundary conditions imposed at infinity to be well-posed.[1]

While a mathematical understanding of the potential non-linear instability mechanisms on aAdS spacetimes seems still out of reach, many results have been obtained for the linear massive wave equation

$$\Box_g \psi + \frac{\alpha}{l^2}\psi = 0, \tag{1}$$

for g an aAdS spacetime and $\alpha < \frac{9}{4}$ the Breitenlohner–Freedman bound[2] imposed on the mass [5].

[1] See [14] for an existence theorem for the full Einstein vacuum equation in this context. In particular, the above instability conjectures have to be supplemented by boundary conditions. In any case, the instability is believed to be present for all boundary conditions which ensure constant (finite) ADM mass at infinity.

[2] Our signature convention will be $(-+++)$ throughout.