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Existence of Klyachko models for $GL(n, \mathbb{R})$ and $GL(n, \mathbb{C})$

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Abstract

We prove that any irreducible unitary representation of $GL(n, \mathbb{R})$ and $GL(n, \mathbb{C})$ admits an equivariant linear form with respect to one of the subgroups considered by Klyachko.

Keywords: Distinguished representations; Unitary dual; Highest derivatives; Mixed models

Contents

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1. Introduction

Let F be either $\mathbb{R}$ or $\mathbb{C}$ and $G_n := GL(n, F)$. For any decomposition $n = r + 2k$ we consider a subgroup of G_n defined by

$$H_{r,2k} = \left\{ \begin{pmatrix} u & X \\ 0 & h \end{pmatrix} \in G_n\colon\ u \in N_r,\ X \in M_{r\times 2k}(F) \text{ and } h \in Sp(2k) \right\}.$$

Here $N_r \subset G_r$ denotes the group of $r \times r$ upper unitriangular matrices and

$$Sp(2k) = \left\{ g \in G_{2k}\colon\ {}^t g J_k g = J_k \right\} \quad \text{where } J_k = \begin{pmatrix} & w_k \\ -w_k & \end{pmatrix} \tag{1}$$

and $w_k \in G_k$ is the permutation matrix with (i, j)-th entry equal to $\delta_{k+1-i,j}$. Let ψ be a non-trivial additive character of F. We associate to ψ the character ψ_r of N_r defined by

$$\psi_r(u) = \psi(u_{1,2} + \cdots + u_{r-1,r})$$

and the character $\phi_{r,2k}$ of $H_{r,2k}$ defined by

$$\phi_{r,2k} \begin{pmatrix} u & X \\ 0 & h \end{pmatrix} = \psi_r(u).$$

Let $\widehat{G_n}$ denote the unitary dual of G_n. For $\pi \in \widehat{G_n}$ we consider the space $\mathrm{Hom}_{H_{r,2k}}(\pi^\infty, \phi_{r,2k})$ of continuous $(H_{r,2k}, \phi_{r,2k})$-equivariant linear forms on the Frechét space π^∞ of smooth vectors in π. We refer to a non-zero element of $\mathrm{Hom}_{H_{r,2k}}(\pi^\infty, \phi_{r,2k})$ as a *Klyachko linear form* of type $(r, 2k)$. Let

$$\mathcal{M}_{r,2k} = \left\{ f : G_n \to \mathbb{C}\colon\ f \text{ is smooth and } f(hg) = \phi_{r,2k}(h) f(g),\ h \in H_{r,2k},\ g \in G_n \right\}.$$

If π is an irreducible Hilbert representation of G_n then a non-zero element $\ell \in \mathrm{Hom}_{H_{r,2k}}(\pi^\infty, \phi_{r,2k})$ defines a realization of π^∞ in the space of functions $\mathcal{M}_{r,2k}$ via $v \mapsto f_v : \pi^\infty \to \mathcal{M}_{r,2k}$ where $f_v(g) = \ell(\pi(g)v)$, $g \in G_n$. We therefore refer to $\mathcal{M}_{r,2k}$ as the *Klyachko model* of type $(r, 2k)$. With this relation in mind for the rest of this paper we focus on Klyachko linear forms rather than Klyachko models.

In order to formulate our main result we recall that the partition $\mathcal{V}(\pi)$, the $SL(2)$-type of π, is defined in [34, Section 2.2] for every $\pi \in \widehat{G_n}$ based on the classification of $\widehat{G_n}$. (See Section 2.4 below.)