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The large-N limit of the Segal–Bargmann transform on $\mathbb{U}_N$

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Abstract

We study the (two-parameter) Segal–Bargmann transform $\mathbf{B}_{s,t}^N$ on the unitary group $\mathbb{U}_N$, for large N. Acting on matrix-valued functions that are equivariant under the adjoint action of the group, the transform has a meaningful limit $\mathscr{G}_{s,t}$ as $N \to \infty$, which can be identified as an operator on the space of complex Laurent polynomials. We introduce the space of *trace polynomials*, and use it to give effective computational methods to determine the action of the heat operator, and thus the Segal–Bargmann transform. We prove several concentration of measure and limit theorems, giving a direct connection from the finite-dimensional transform $\mathbf{B}_{s,t}^N$ to its limit $\mathscr{G}_{s,t}$. We characterize the operator $\mathscr{G}_{s,t}$ through its inverse action on the standard polynomial basis. Finally, we show that, in the case $s = t$, the limit transform $\mathscr{G}_{t,t}$ is the "free Hall transform" $\mathscr{G}^t$ introduced by Biane.

Keywords: Segal–Bargmann transform; Heat kernel analysis on Lie groups

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Contents

1. Introduction

The *Segal–Bargmann transform* (also known in the physics literature as the *Bargmann transform* or *coherent state transform*) is a unitary isomorphism from L^2 to holomorphic L^2. It was originally introduced by Segal [30–32] and Bargmann [1,2], as a map

$$S_t : L^2(\mathbb{R}^N, \gamma_t^N) \to \mathcal{H}L^2(\mathbb{C}^N, \gamma_{t/2}^{2N})$$

where γ_t^N is the standard Gaussian heat kernel measure $(\frac{1}{2\pi t})^{N/2} \exp(-\frac{1}{2t}|\mathbf{x}|^2)\, d\mathbf{x}$ on $\mathbb{R}^N$, and $\mathcal{H}L^2$ denotes the subspace of square-integrable *holomorphic* functions. The transform S_t is given by convolution with the heat kernel, followed by analytic continuation.

In [17], the second author introduced an analog of the Segal–Bargmann transform for any compact Lie group K. Let Δ_K denote the Laplace operator over K (determined by an Ad-invariant inner product on the Lie algebra $\mathfrak{k}$ of K), and denote by $e^{\frac{t}{2}\Delta_K}$ the corresponding heat operator. The generalized Segal–Bargmann transform B_t maps functions on K to holomorphic functions on the complexification $K_{\mathbb{C}}$ of K, by application of the heat operator and analytic continuation.