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Inner functions on the bidisk and associated Hilbert spaces

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Abstract

Matrix valued inner functions on the bidisk have a number of natural subspaces of the Hardy space on the torus associated to them. We study their relationship to Agler decompositions, regularity up to the boundary, and restriction maps into one variable spaces. We give a complete description of the important spaces associated to matrix rational inner functions. The dimension of some of these spaces can be computed in a straightforward way, and this ends up having an application to the study of three variable rational inner functions. Examples are included to highlight the differences between the scalar and matrix cases.

Keywords: Inner function; Bidisk; Polydisk; Bidisc; Polydisc; Hardy space; Agler decomposition

Contents

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1. Introduction

Inner functions ϕ on the unit disk $\mathbb{D}$, their associated Hilbert spaces ϕH^2 and $\mathcal{H}_\phi = H^2 \ominus \phi H^2$, and the associated shift S and backward shift S^* operators on these spaces form a natural and rich area of analysis. Natural, because by Beurling's theorem [11], every invariant subspace of the forward shift on $\ell^2(\mathbb{N})$ is unitarily equivalent to ϕH^2 for some inner ϕ. Rich, because if we allow ϕ to be operator valued, then any contractive operator on a separable Hilbert space can be modeled as S^* on $\mathcal{H}_\phi$ for some ϕ. At the same time, simple choices for ϕ provide interesting examples. If ϕ is a finite Blaschke product, the space $\mathcal{H}_\phi$ is finite dimensional and is related to orthogonal polynomials on the unit circle. If $\phi(z) = \exp(a\frac{z+1}{z-1})$, $a > 0$, then $\mathcal{H}_\phi$ is isometric to the Paley–Wiener space $PW_a = \mathcal{F}(L^2(0,a))$ through a change of coordinates to the upper half-plane. See [18] or [28] for the one variable theory.

Inner functions on the bidisk $\mathbb{D}^2 = \mathbb{D} \times \mathbb{D}$ and their associated Hilbert spaces are far richer and considerably less well developed than their one variable counterparts. For early work on the topic see for instance Rudin [29], Ahern and Clark [8], Ahern [7], and Sawyer [31]. Rational inner functions on the bidisk have close ties to the study of stable bivariate polynomials (e.g. polynomials with no zeros on the bidisk), and Hilbert space methods have proved useful in understanding them. See Cole and Wermer [13], Geronimo and Woerdeman [15], Ball, Sadosky and Vinnikov [10], Woerdeman [32], Knese [22], and Geronimo, Iliev and Knese [16]. Any type of general classification of inner functions on the bidisk or polydisk seems unknown and difficult.

Recall that $\phi : \mathbb{D}^2 \to \overline{\mathbb{D}}$ is an inner function if ϕ is holomorphic and satisfies

$$\lim_{r \nearrow 1} \left|\phi\left(re^{i\theta_1}, re^{i\theta_2}\right)\right| = \left|\phi\left(e^{i\theta_1}, e^{i\theta_2}\right)\right| = 1 \quad \text{a.e.}$$

We also use the term inner function for holomorphic functions $\phi : \mathbb{D}^2 \to \mathcal{B}_1$, where $\mathcal{B}_1$ is the closed unit ball in the operator norm of the bounded linear operators from a separable Hilbert space $\mathcal{V}$ to itself such that

$$\phi(z)^*\phi(z) = \phi(z)\phi(z)^* = I \quad \text{for a.e. } z \in \mathbb{T}^2 = (\partial\mathbb{D})^2,$$

i.e. ϕ is unitary valued almost everywhere on the torus. Note that the radial boundary limits of these operator valued functions converge in the strong operator topology:

$$\lim_{r \nearrow 1} \phi\left(re^{i\theta_1}, re^{i\theta_2}\right)v = \phi\left(e^{i\theta_1}, e^{i\theta_2}\right)v$$

for each $v \in \mathcal{V}$ and for a.e. $(\theta_1, \theta_2) \in [0, 2\pi]^2$.