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The spectrum of differential operators and square-integrable solutions ☆

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Abstract

We give a comprehensive account of the relationship between the square-integrable solutions for real values of the spectral parameter λ and the spectrum of self-adjoint even order ordinary differential operators with real coefficients and arbitrary deficiency index d and we solve an open problem stated by Weidmann in his well-known 1987 monograph. According to a well-known result, if one endpoint is regular and for some real value of the spectral parameter λ the number of linearly independent square-integrable solutions is less than d, then λ is in the essential spectrum of every self-adjoint realization of the equation. Weidmann extends this result to the two singular endpoint case provided an additional condition is satisfied. Here we prove this result without the additional condition.

Keywords: Differential operators; Continuous spectrum; Deficiency index; Singular boundary conditions

1. Introduction

The spectrum of a self-adjoint ordinary differential operator in Hilbert space $H = L^2(J, w)$, $J = (a, b)$, is real, consists of eigenvalues of finite multiplicity, of essential and of continuous spectrum. A number λ is an eigenvalue if the corresponding differential equation has a nontrivial solution which satisfies the boundary conditions. On the other hand, the essential spectrum is

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independent of the boundary conditions and thus depends only on the coefficients, including the weight function w, of the equation. This dependence is implicit and highly complicated. The coefficients and the weight function also determine the deficiency index d of the minimal operator $S_{\min}$ determined by the equation. For real-valued coefficients this is the number of linearly independent solutions in H for nonreal values of the spectral parameter λ and this number is independent of λ provided $\operatorname{Im}(\lambda) \neq 0$. For real values of λ the number of linearly independent solutions $r(\lambda)$ in H varies with λ. It is this dependence on λ which we exploit to get information about the spectrum. For instance we settle an open problem of Weidmann for the class of operators studied here, see Theorems 3 and 5 below.

The contrasting behavior of $r(\lambda)$ for the two singular endpoint case from the one singular endpoint case has some interesting consequences. In the case of only one singular endpoint we have $r(\lambda) \leqslant d$ whereas in the two singular endpoint case $r(\lambda)$ may assume values less than d, equal to d, or greater then d. The case $r(\lambda) > d$ leads to the surprising result that this value of λ is an eigenvalue of every self-adjoint extension, i.e. for any given self-adjoint boundary condition, there are eigenfunctions of λ which satisfy this boundary condition.

Remark 1. We comment on our approach using knowledge of the number of square-integrable solutions for *real values of the spectral parameter* λ to obtain information about the spectrum. This approach contrasts with some commonly used methods such as asymptotic approximations of solutions and perturbation theory, see [27,18,14,6,20,17,1–4,8,7,19] and the references in these books and papers.

2. Definitions and preliminary results

We study spectral properties of the self-adjoint realizations of the equation

$$My = \lambda w y \tag{2.1}$$

on the intervals

$$J = (a, b), \qquad J_a = (a, c), \quad \text{and} \quad J_b = (c, b), \quad -\infty \leqslant a < c < b \leqslant \infty,$$

in the Hilbert spaces $H = L^2(J, w)$, $H_a = L^2(J_a, w)$, $H_b = L^2(J_b, w)$, respectively, where M is a general symmetric quasi-differential expression of order $n = 2k$, $k \geqslant 1$ (to be defined below) with real valued coefficients, $w \in L_{loc}(J)$, $w > 0$ on J, each endpoint a, b may be regular or singular.

For sufficiently smooth real valued coefficients, the most general symmetric (formally self-adjoint) differential expressions M of order $n = 2k$, $k \geqslant 1$, have the form [6,20],

$$My = \sum_{j=0}^{k} \big(p_j y^{(j)}\big)^{(j)}. \tag{2.2}$$

We are interested in using much weaker conditions, i.e., local Lebesgue integrability, on the coefficients. For this purpose Eq. (2.2) is modified by using quasi-derivatives $y^{[j]}$ as follows: