Available online at www.sciencedirect.com
SciVerse ScienceDirect
Journal of Functional Analysis 263 (2012) 2832–2862
JOURNAL OF Functional Analysis
www.elsevier.com/locate/jfa

Energy concentration and explicit Sommerfeld radiation condition for the electromagnetic Helmholtz equation

Miren Zubeldia [1]

Universidad del País Vasco, Departamento de Matemáticas, Apartado 644, 48080, Spain

Received 5 January 2012; accepted 14 August 2012
Available online 31 August 2012
Communicated by I. Rodnianski

Abstract

We study the electromagnetic Helmholtz equation

$$\big(\nabla + ib(x)\big)^2 u(x) + n(x)u(x) = f(x), \quad x \in \mathbb{R}^d,$$

with the magnetic vector potential $b(x)$ and $n(x)$ a variable index of refraction that does not necessarily converge to a constant at infinity, but can have an angular dependence like $n(x) \to n_\infty(\frac{x}{|x|})$ as $|x| \to \infty$. We prove an explicit Sommerfeld radiation condition

$$\int_{\mathbb{R}^d} \left| \nabla_b u - i n_\infty^{1/2} \frac{x}{|x|} u \right|^2 \frac{dx}{1+|x|} < +\infty$$

for solutions obtained from the limiting absorption principle and we also give a new energy estimate

$$\int_{\mathbb{R}^d} \left| \nabla_\omega n_\infty \left(\frac{x}{|x|} \right) \right|^2 \frac{|u|^2}{1+|x|} \, dx < +\infty,$$

which explains the main physical effect of the angular dependence of n at infinity and deduces that the energy concentrates in the directions given by the critical points of the potential.

E-mail address: miren.zubeldia@ehu.es.
[1] The author is partially supported by the Spanish grant FPU AP2007-02659 of the MEC, by Spanish Grant MTM2007-62186 and by the ERC Advanced Grant – Mathematical foundations (ERC-AG-PE1) of the European Research Council.

0022-1236/$ – see front matter © 2012 Elsevier Inc. All rights reserved.
http://dx.doi.org/10.1016/j.jfa.2012.08.014

Keywords: Magnetic potential; Helmholtz equation; Sommerfeld condition; Energy concentration

1. Introduction

Let us consider the following Helmholtz equation

$$\big(\nabla + ib(x)\big)^2 u(x) + n(x)u(x) = f(x), \quad x \in \mathbb{R}^d, \tag{1.1}$$

where $b = (b_1, \dots, b_d) : \mathbb{R}^d \to \mathbb{R}^d$ is a magnetic potential and $n : \mathbb{R}^d \to \mathbb{R}$ is a variable index of refraction that admits a radial limit

$$n(x) \to n_\infty\left(\frac{x}{|x|}\right) \quad \text{as } |x| \to \infty. \tag{1.2}$$

In this work we are interested in the study of the existence and uniqueness of solution of Eq. (1.1) with an appropriate radiation condition using the limiting absorption method as well as some new estimates that characterize the behavior of the solution at infinity.

The radiation conditions are known to be necessary for the uniqueness of solutions to (1.1), as first remarked by Sommerfeld [32] in the free case $b \equiv 0 \equiv V$. Moreover, there is a strong connection between this topic and the limiting absorption principle for Schrödinger operators, to which a lot of research has been devoted (see, for example, [6,15,17,18,1,2,14,29,21–23,30,16,4,7,31,26,28,33]). Let us now give a brief picture about the recent advances on Sommerfeld radiation.

As a first result we mention Eidus [6], where it is showed that there exists a unique solution $u(\lambda, f)$ of Eq. (1.1) with $n(x) = \lambda + V(x)$ in $\mathbb{R}^3$ satisfying the radiation condition

$$\lim_{r\to\infty} \int_{|x|=r} \left|\frac{\partial u}{\partial |x|} - i\lambda^{1/2} u\right|^2 d\sigma(r) = 0. \tag{1.3}$$

Here $b_j(x)$ is assumed to vanish close to infinity and the electric potential satisfies $V(x) = O(|x|^{-2-\alpha})$ with $\alpha > \frac{1}{6}$ at infinity. In 1972, Ikebe and Saito [15] extended the result by Eidus to electric potentials of the form $V = \lambda\tilde{p} + Q$, where $\tilde{p}$ is long range and Q is short range, obtaining the precise radiation condition

$$\int_{\mathbb{R}^d} \left|(\nabla + ib)u - i\lambda^{1/2}\frac{x}{|x|}u\right|^2 \frac{dx}{(1+|x|)^{1-\delta}} < +\infty, \tag{1.4}$$

where $0 < \delta < 1$ is a fixed constant. This result has been recently improved by Zubeldia in [33], by adding some singularities on the potentials at the origin and extending the range of δ to $0 < \delta < 2$, in (1.4). In the purely electric case $b \equiv 0$, under similar assumptions on V, Saito proved in [29,30] another type of radiation condition,

$$\int_{\mathbb{R}^d} \left|\nabla u - i(\nabla K)u\right|^2 \frac{dx}{(1+|x|)^{1-\delta}} < +\infty,$$