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# Grothendieck's nuclear operator theorem revisited with an application to $p$-null sequences ☆

Eve Oja [a,b,*]

[a] *Faculty of Mathematics and Computer Science, Tartu University, J. Liivi 2, 50409 Tartu, Estonia*
[b] *Estonian Academy of Sciences, Kohtu 6, 10130 Tallinn, Estonia*

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**Abstract**

Let $X$ and $Y$ be Banach spaces and let $\alpha$ be a tensor norm. The principal result is the following theorem. If either $X^{***}$ or $Y$ has the approximation property, then each $\alpha$-nuclear operator $T: X^* \to Y$ such that $T^*(Y^*) \subset X$ can be approximated in the $\alpha$-nuclear norm by finite-rank operators of type $X \otimes Y$. In the special case of (Grothendieck) nuclear operators, the theorem provides a strengthening for the classical theorem on the nuclearity of operators with a nuclear adjoint. The hypotheses about the approximation property are essential. The main application yields an affirmative answer to [C. Piñeiro, J.M. Delgado, $p$-Convergent sequences and Banach spaces in which $p$-compact sets are $q$-compact, Proc. Amer. Math. Soc. 139 (2011) 957–967]: for $p \geqslant 1$, a sequence $(x_n) \subset X$ is $p$-null if and only if $\lim x_n = 0$ and $(x_n)$ is relatively $p$-compact in $X$.


*Keywords:* Banach spaces; Nuclear operators; Approximation property; Tensor products; $\alpha$-Nuclear operators; Operator ideals; $p$-Compactness; $p$-Null sequences

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* Correspondence to: Faculty of Mathematics and Computer Science, Tartu University, J. Liivi 2, 50409 Tartu, Estonia.
*E-mail address:* eve.oja@ut.ee.

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## 1. Introduction

Let $X$ be a Banach space and let $p \geqslant 1$ be a real number. Recently, Delgado and Piñeiro [25] introduced and studied an interesting class $c_{0,p}(X)$ of $p$-null sequences that sits, as a linear subspace, in $c_0(X)$, the space of $X$-valued null sequences. The following question was asked in [25] (see Section 4.1 below for the relevant terminology).

**Question 1.1** *(Delgado–Piñeiro).* Is a sequence $(x_n) \in c_0(X)$ $p$-null if and only if $(x_n)$ is relatively $p$-compact in $X$?

In the final Section 4, we shall show, after representing $c_{0,p}(X)$ as the Chevet–Saphar tensor product $c_0 \hat{\otimes}_{d_p} X$ (see Theorem 4.1), that the answer to Question 1.1 is affirmative (see Theorem 4.3). It turns out that, surprisingly enough, Question 1.1 essentially reduces to a structural problem concerning $\alpha$-nuclear operators, where $\alpha$ is a tensor norm (for the terminology, see Section 2.1). For convenience, we shall state this problem in the case of (classical) nuclear operators as Question 1.2 below. To formulate Question 1.2, we need to recall some standard notation.

Let $X$ and $Y$ be Banach spaces. A bounded linear operator $T \in \mathcal{L}(X,Y)$ is said to be *nuclear* if there exist $x_n^* \in X^*$ and $y_n \in Y$ such that $\sum_{n=1}^{\infty} \|x_n^*\| \|y_n\| < \infty$ and $Tx = \sum_{n=1}^{\infty} x_n^*(x) y_n$ for all $x \in X$. In this case, one writes $T = \sum_{n=1}^{\infty} x_n^* \otimes y_n$ and calls the latter sum a *nuclear representation* of $T$. Let us denote by $\mathcal{N}(X,Y)$ the collection of all nuclear operators from $X$ to $Y$.

**Question 1.2.** Let $T \in \mathcal{N}(X^*,Y)$ satisfy $T^*(Y^*) \subset X$. Does $T$ admit a nuclear representation $T = \sum_{n=1}^{\infty} x_n \otimes y_n$ with $x_n \in X$ and $y_n \in Y$?

In Section 2 we study Question 1.2 in a more general context of $\alpha$-nuclear operators, since this context appears to be necessary for solving Question 1.1. The principal result of the present paper is Theorem 2.4. It is proved and applied to $\alpha$-nuclear operators in Section 3 and to $p$-null sequences in Section 4. Let us point out here a relevant special case of Theorem 2.4.

**Theorem 1.3.** *If either $X^{***}$ or $Y$ has the approximation property, then the answer to Question* 1.2 *is affirmative.*

Theorem 1.3 is actually a special case of Theorem 3.1 for the $p$-nuclear and right $p$-nuclear operators $\mathcal{N}_p$ and $\mathcal{N}^p$ (note that $\mathcal{N} = \mathcal{N}_1 = \mathcal{N}^1$). And this is Theorem 3.1 which will be used in our main application (in Section 4) showing that the answer to Question 1.1 is affirmative. Curiously enough, we shall need the case when $X^{***}$ has the approximation property.

Theorem 1.3 represents a (new) contribution to Grothendieck's classics on nuclear operators. It appears to be a stronger result than the well-known theorem on operators with a nuclear adjoint.

**Theorem 1.4** *(Grothendieck–Oja–Reinov). Assume that either $X^*$ or $Y^{***}$ has the approximation property. If $T \in \mathcal{L}(X,Y)$ and $T^* \in \mathcal{N}(Y^*,X^*)$, then $T \in \mathcal{N}(X,Y)$.*

(Recall that Theorem 1.4 was proved in [10, Chapter I, pp. 85–86] under the hypothesis on $X$ (see, e.g., [32, Proposition 4.10]) and in [22] under the hypothesis on $Y$ (announced in [21]); see [17] for a simpler proof in the both cases.)