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Existence result for a class of semilinear totally characteristic hypoelliptic equations with conical degeneration

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Abstract

In this article, we use the cone Sobolev inequality and the Poincaré inequality to prove the existence theorem for a class of semilinear degenerate hypoelliptic equation on manifolds with conical singularities. In this paper we shall find the existence theorem for the problem (1.1) in cone Sobolev space $\mathcal{H}_{2,0}^{1,\frac{n}{2}}(\mathbb{B})$. Finally, we obtain existence result of global solutions with exponential decay and show the blow-up in finite time of solutions.

Keywords: Semilinear equations; Totally characteristic degeneracy; Global solution; Blow-up

1. Introduction

In this paper, we use the cone Sobolev inequality and the Poincaré inequality to prove the existence theorem for a class of semilinear degenerate hypoelliptic pseudo-differential equation [1] and [2], on manifolds with conical singularities. More precisely, we study the following initial–boundary value problem for a class of semilinear pseudo-differential equations

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$$\begin{cases} \partial_t^k u - \Delta_{\mathbb{B}} u + V(x)u = g(x)|u|^{p-1}u, & x \in int\mathbb{B},\ t > 0, \\ u(0,x) = u_0(x), & x \in int\mathbb{B}, \\ \partial_t^{k-1} u(t,x) = 0, & x \in int\mathbb{B},\ t \geqslant 0,\ k \geqslant 1, \end{cases} \tag{1.1}$$

where, $2 < p+1 < \frac{2n}{n-2} = 2^*$ is the critical cone Sobolev exponents. Here the domain $\mathbb{B}$ is $[0,1) \times X$, X is an $(n-1)$-dimensional closed compact manifold, which is regarded as the local model near the conical points on manifolds with conical singularities, and $\partial\mathbb{B} = \{0\} \times X$. Moreover, the operator $\Delta_{\mathbb{B}}$ in (1.1) is defined by $(x_1\partial_{x_1})^2 + \partial_{x_2}^2 + \cdots + \partial_{x_n}^2$, which is an elliptic operator with totally characteristic degeneracy on the boundary $x_1 = 0$, we also call it Fuchsian type Laplace operator, and the corresponding gradient operator by $\nabla_{\mathbb{B}} := (x_1\partial_{x_1}, \partial_{x_2}, \dots, \partial_{x_n})$. Near $\partial\mathbb{B}$ we will often use coordinates $(x_1, x) = (x_1, x_2, \dots, x_n)$ for $x_1 \in [0,1)$ and $x \in X$. In this paper we shall find the existence theorem for the problem (1.1) in cone Sobolev space $\mathcal{H}_{2,0}^{1,\frac{n}{2}}(\mathbb{B})$ which will be given in the next section. In Eq. (1.1), we assume that $V(x) \in L^\infty(int\mathbb{B}) \cap C(int\mathbb{B})$ is positive potential function such that $(C^*)^2 \neq 1$ where,

$$C^* = \sup\left\{ \frac{\|\sqrt{V(x)}u(x)\|_{L_2^{\frac{n}{2},k}(\mathbb{B})}}{\|\nabla_{\mathbb{B}} u\|_{L_2^{\frac{n}{2},k}(\mathbb{B})}};\ u \in \mathcal{H}_{2,0}^{1,\frac{n}{2}}(\mathbb{B}) \right\}.$$

For finding such positive potential function any one can consider Poincaré's constant on manifold $\mathbb{B}$. The function $g \in L^\infty(int\mathbb{B}) \cap C(int\mathbb{B})$ is a non-negative weighted function. Our study is in fact motivated by the study of [12] and we shall apply potential method which was established by Sattinger [15]. So based on Schrohe and Seiler's cone Sobolev algebra [16], we study the existence and non-existence global weak solutions for higher-order semilinear partial differential equations with respect to variable time with positive potential function and a non-negative weighted function. The well-known operator $\Delta_{\mathbb{B}} + V(x)$ (see [9]) appears naturally in the nonlinear heat and wave equations [14], nonlinear Schrödinger equation with potential function [10] and the Wheeler–De Witt equation which deals with the minisuperspace model in quantum cosmology that we can refer the reader to [18] and the references therein for a complete description of the model. Our problem can be seen then as a higher-order evolution version of the nonlinear hypoelliptic pseudo-differential equations for which in the case that $k = 1$, $V(x) = 0$ and $g(x) \equiv 1$ then the problem (1.1) is reduced to problem (1.1) in [5] and in the classical sense our problem include the classical problem

$$\begin{cases} \partial_t u - \Delta u = |u|^{p-1}u, & x \in \Omega,\ t > 0, \\ u(0,x) = u_0(x), & x \in \Omega, \\ u(t,x) = 0, & x \in \Omega,\ t \geqslant 0, \end{cases} \tag{1.2}$$

where Ω is bounded domain of $\mathbb{R}^n$ with smooth boundary $\partial\Omega$ and Δ is the standard Laplace operator. Its well-known that problem (1.2) has been studied by many authors [7,17,19]. In the case $k = 2$ we get the nonlinear Schrödinger equation with initial–Neumann boundary condition which this type of equations had been studied by many authors [6,10,9].

In this paper, we shall consider the corresponding problem (1.1) on the manifold with conical singularities. Similar to the classical case, we introduced the following functionals on cone Sobolev space $\mathcal{H}_{2,0}^{1,\frac{n}{2}}(\mathbb{B})$: