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Berezin transforms on noncommutative varieties in polydomains ☆

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Abstract

Let $\mathcal{Q}$ be a set of polynomials in noncommutative indeterminates $Z_{i,j}$, $i \in \{1, \ldots, k\}$, $j \in \{1, \ldots, n_i\}$. In this paper, we study noncommutative varieties

$$\mathcal{V}_{\mathcal{Q}}(\mathcal{H}) := \big\{\mathbf{X} = \{X_{i,j}\} \in \mathbf{D}(\mathcal{H}):\ g(\mathbf{X}) = 0 \text{ for all } g \in \mathcal{Q}\big\},$$

where $\mathbf{D}(\mathcal{H})$ is a *regular polydomain* in $B(\mathcal{H})^{n_1+\cdots+n_k}$ and $B(\mathcal{H})$ is the algebra of bounded linear operators on a Hilbert space $\mathcal{H}$. Under natural conditions on $\mathcal{Q}$, we show that there is a *universal model* $\mathbf{S} = \{\mathbf{S}_{i,j}\}$ such that $g(\mathbf{S}) = 0$, $g \in \mathcal{Q}$, acting on a subspace of a tensor product of full Fock spaces. We characterize the variety $\mathcal{V}_{\mathcal{Q}}(\mathcal{H})$ and its pure part in terms of the universal model and a class of completely positive linear maps. We obtain a characterization of those elements in $\mathcal{V}_{\mathcal{Q}}(\mathcal{H})$ which admit characteristic functions and prove that the characteristic function is a complete unitary invariant for the class of completely non-coisometric elements. We study the universal model $\mathbf{S}$, its joint invariant subspaces and the representations of the universal operator algebras it generates: the *variety algebra* $\mathcal{A}(\mathcal{V}_{\mathcal{Q}})$, the Hardy algebra $F^\infty(\mathcal{V}_{\mathcal{Q}})$, and the C^*-algebra $C^*(\mathcal{V}_{\mathcal{Q}})$. Using noncommutative Berezin transforms associated with each variety, we develop an operator model theory and dilation theory for large classes of varieties in noncommutative polydomains. This includes various commutative cases which are closely connected to the theory of holomorphic functions in several complex variables and algebraic geometry.

Keywords: Multivariable operator theory; Berezin transform; Noncommutative polydomain; Noncommutative variety; Free holomorphic function; Fock space; Invariant subspace; Dilation theory; Characteristic function

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0. Introduction

We denote by $B(\mathcal{H})^{n_1} \times_c \cdots \times_c B(\mathcal{H})^{n_k}$ the set of all tuples $\mathbf{X} := (X_1, \ldots, X_k)$ in $B(\mathcal{H})^{n_1} \times \cdots \times B(\mathcal{H})^{n_k}$ with the property that the entries of $X_s := (X_{s,1}, \ldots, X_{s,n_s})$ are commuting with the entries of $X_t := (X_{t,1}, \ldots, X_{t,n_t})$ for any $s, t \in \{1, \ldots, k\}$, $s \neq t$. In an attempt to unify the multivariable operator model theory for the ball-like domains and commutative polydiscs, we developed in [33] an operator model theory and a theory of free holomorphic functions on *regular polydomains* of the form

$$\mathbf{D}_{\mathbf{q}}^{\mathbf{m}}(\mathcal{H}) := \left\{\mathbf{X} = (X_1, \ldots, X_k) \in B(\mathcal{H})^{n_1} \times_c \cdots \times_c B(\mathcal{H})^{n_k}:\ \mathbf{\Delta}_{\mathbf{q},\mathbf{X}}^{\mathbf{p}}(I) \geqslant 0 \text{ for } \mathbf{0} \leqslant \mathbf{p} \leqslant \mathbf{m}\right\},$$

where $\mathbf{m} := (m_1, \ldots, m_k)$ and $\mathbf{n} := (n_1, \ldots, n_k)$ are in $\mathbb{N}^k$, the *defect mapping* $\mathbf{\Delta}_{\mathbf{q},\mathbf{X}}^{\mathbf{m}} : B(\mathcal{H}) \to B(\mathcal{H})$ is defined by

$$\mathbf{\Delta}_{\mathbf{q},\mathbf{X}}^{\mathbf{m}} := (id - \Phi_{q_1,X_1})^{m_1} \circ \cdots \circ (id - \Phi_{q_k,X_k})^{m_k},$$

and $\mathbf{q} = (q_1, \ldots, q_k)$ is a k-tuple of positive regular polynomials $q_i \in \mathbb{C}[Z_{i,1}, \ldots, Z_{i,n_i}]$, i.e., all the coefficients of q_i are positive, the constant term is zero, and the coefficients of the linear terms $Z_{i,1}, \ldots, Z_{i,n_i}$ are different from zero. If the polynomial q_i has the form $q_i = \sum_\alpha a_{i,\alpha} Z_{i,\alpha}$, the completely positive linear map $\Phi_{q_i,X_i} : B(\mathcal{H}) \to B(\mathcal{H})$ is defined by setting $\Phi_{q_i,X_i}(Y) := \sum_\alpha a_{i,\alpha} X_{i,\alpha} Y X_{i,\alpha}^*$ for $Y \in B(\mathcal{H})$.

In this paper, we study noncommutative varieties in the polydomain $\mathbf{D}_{\mathbf{q}}^{\mathbf{m}}(\mathcal{H})$, given by

$$\mathcal{V}_{\mathcal{Q}}(\mathcal{H}) := \left\{\mathbf{X} \in \mathbf{D}_{\mathbf{q}}^{\mathbf{m}}(\mathcal{H}):\ g(\mathbf{X}) = 0 \text{ for all } g \in \mathcal{Q}\right\},$$

where $\mathcal{Q}$ is a set of polynomials in noncommutative indeterminates $Z_{i,j}$, which generates a nontrivial ideal in $\mathbb{C}[Z_{i,j}]$. The goal is to understand the structure of this noncommutative variety, determine its elements and classify them up to unitary equivalence, for large classes of sets $\mathcal{Q} \subset \mathbb{C}[Z_{i,j}]$. This study can be seen as an attempt to initiate noncommutative algebraic geometry in polydomains.

To present our results, we need some notation. Let H_{n_i} be an n_i-dimensional complex Hilbert space. We consider the full Fock space of H_{n_i} defined by

$$F^2(H_{n_i}) := \bigoplus_{p \geqslant 0} H_{n_i}^{\otimes p},$$