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Localization and dimension free estimates for maximal functions

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Abstract

In the recent paper [A. Naor, T. Tao, Random martingales and localization of maximal inequalities, J. Funct. Anal. 259 (3) (2010) 731–779], Naor and Tao introduce a new class of measures with a so-called micro-doubling property and present, via martingale theory and probability methods, a localization theorem for the associated maximal functions. As a consequence they obtain a weak type estimate in a general abstract setting for these maximal functions that is reminiscent of the '$n \log n$ result' of Stein and Strömberg in Euclidean spaces. The purpose of this work is twofold. First we introduce a new localization principle that localizes not only in the time-dilation parameter but also in space. The proof uses standard covering lemmas and selection processes. Second, we show that a uniform condition for micro-doubling in the Euclidean spaces provides dimension free estimates for their maximal functions in all L^p with $p > 1$. This is done introducing a new technique that allows to differentiate through dimensions.

Keywords: Maximal functions; Localization; Micro-doubling properties; Dimension free estimates

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1. Introduction

We say that (X, d, μ) is a metric measure space if (X, d) is a separable metric space and μ a Radon measure on it. We denote by $B(x, r)$ the open ball centered at x with radius r with respect to the metric d, that is

$$B(x, r) = \{y \in X\colon\ d(x, y) < r\}.$$

We will assume that the measure μ is non-degenerate. This means that any ball with positive radius has non-zero measure. Given $T \subset (0, \infty)$, for a locally integrable function f over X we define the following centered maximal operator

$$M_T f(x) = \sup_{r\in T} \frac{1}{\mu(B(x, r))} \int_{B(x,r)} |f(y)|\, d\mu(y).$$

When $T = (0, \infty)$, the operator $M = M_T$ represents the usual Hardy–Littlewood maximal operator. As in the case of evolution equations and semigroup theory we can think of T as a set of times and then M_T is the maximal operator on them.

We will also assume that the measure μ satisfies a doubling property. This simply says that the measure of a ball is comparable with the measure of certain dilation of it. The classical way of expressing the doubling property uses the dilation factor 2. That is, μ is doubling if there exists a constant $K > 0$ so that for each $x \in X$ and $R > 0$ one has $\mu(B(x, 2R)) \leqslant K\mu(B(x, R))$.

In these hypotheses one can reproduce the argument of Vitali's covering lemma to show that M_T is weakly bounded on $L^1(\mu)$. That is, there exists a constant $c_{\mu,1} > 0$ so that

$$\mu(\{x \in X\colon\ M_T f(x) > \lambda\}) \leqslant \frac{c_{\mu,1}}{\lambda} \int_X |f|\, d\mu,$$

for all $\lambda > 0$ and all locally integrable f over X. Since $\|M_T f\|_{L^\infty(\mu)} \leqslant \|f\|_{L^\infty(\mu)}$, by interpolation one obtains the $L^p(\mu)$ bounds

$$\|M_T f\|_{L^p(\mu)} \leqslant C_{\mu,p}\|f\|_{L^p(\mu)},$$

for all $p > 1$. The order of magnitude of the constants $c_{\mu,1}$ and $C_{\mu,p}$ that we obtain is essentially that of the doubling constant K and $K^{1/p}p/(p-1)$ respectively.

In the case that $X = \mathbb{R}^n$ and $\mu = m_n$ is Lebesgue measure, the value of the doubling constant K is 2^n. Hence the aforementioned constants $c_{m_n,1}$ and $C_{m_n,p}$ grow exponentially to infinity with the dimension. It has been a matter of interest to know if these constants can be bounded uniformly in dimension. E.M. Stein [27] (detailed proof in [28]) proved this for $C_{m_n,p}$ with $p > 1$ when considering the Euclidean metric. J. Bourgain [6–8] and independently A. Carbery [10] showed that for the metric given by a norm, uniform bounds hold if $p > 3/2$. This was extended to all the range $p > 1$ by D. Müller in [25] for the maximal functions associated with the ℓ^q metrics, with $1 \leqslant q < \infty$. Observe that this excludes the case $q = \infty$, where the 'balls' are cubes with sides parallel to the coordinate axes. Recently J. Bourgain has solved this case by giving uniform bounds on L^p for all $p > 1$ (see [9]).

The case $p = 1$ is more complicated. J.M. Aldaz showed in [2] that for the ℓ^∞ metric the constants $c_{m_n,1}$ grow to infinity as $n \to \infty$. As shown by Aubrun [5], in this case one has the