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Homogenization of surface and length energies for spin systems

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Abstract

We study the homogenization of lattice energies related to Ising systems of the form

$$E_\varepsilon(u) = -\sum_{ij} c_{ij}^\varepsilon u_i u_j,$$

with u_i a spin variable indexed on the portion of a cubic lattice $\Omega \cap \varepsilon\mathbb{Z}^d$, by computing their Γ-limit in the framework of surface energies in a BV setting. We introduce a notion of homogenizability of the system $\{c_{ij}^\varepsilon\}$ that allows to treat periodic, almost-periodic and random statistically homogeneous models (the latter in dimension two), when the coefficients are positive (ferromagnetic energies), in which case the limit energy is finite on $BV(\Omega;\{\pm 1\})$ and takes the form

$$F(u) = \int_{\Omega\cap\partial^*\{u=1\}} \varphi(\nu)\, d\mathcal{H}^{d-1}$$

(ν is the normal to $\partial^*\{u=1\}$), where φ is characterized by an asymptotic formula. In the random case φ can be expressed in terms of first-passage percolation characteristics. The result is extended to coefficients with varying sign, under the assumption that the areas where the energies are antiferromagnetic are well-separated. Finally, we prove a dual result for discrete curves.

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1. Introduction

In this paper we study surface energies defined on lattice systems through bond interactions. These energies are related to Ising energies, commonly written in the form

$$E(u) = -\sum_{ij} c_{ij} u_i u_j,$$

where u_i is a spin variable taking values $+1$ or -1 and i, j are indices varying in (a suitable subset of) a square lattice $\mathbb{Z}^d$ (see e.g. [19]). If the coefficients c_{ij} are supposed to be positive then ground states are constant. Even with this assumption, if boundary conditions or additional constraint are added, minimizers are not trivial and it is interesting to determine their behaviour when the minimization process involves an increasingly large number of indices. This problem can be set in a variational framework involving energies on lattice subsets, after identifying a function $\{u_i\}$ with the set $A = \{i \in \mathbb{Z}^d\colon\ u_i = 1\}$. To that end, note that the energy above can be written equivalently as

$$E(u) = \sum_{ij} c_{ij}(u_j - u_j)^2,$$

upon addition of a constant. Under the simplifying assumption that the relevant interactions are those between nearest neighbours (i.e., that we may assume $c_{ij} = 0$ if $|i - j| \neq 1$) then this energy can be seen as a discrete surface energy, concentrated on the boundary of A; i.e., on nearest-neighbour pairs (i, j) such that $u_i \neq u_j$. If moreover c_{ij} is constant for nearest-neighbour interactions, then we have the prototypical *ferromagnetic energy* of a subset A of the lattice $\mathbb{Z}^d$, defined as

$$E(A) = \#\big\{(i, j)\colon\ i \in A,\ j \notin A,\ |i - j| = 1\big\}.$$

A continuous approximation of such energies can be obtained in the framework of surface energies defined on sets of finite perimeter. In fact, if we identify each lattice subset A with the union of (coordinate) cubes $\bigcup_{i\in A}(i + Q)$, where $Q = (-1/2, 1/2)^d$ is the unit coordinate cube centered in 0, and we still denote this set by A with a slight abuse of notation, then

$$E(A) = \mathcal{H}^{d-1}(\partial A)$$

coincides with the $(d - 1)$-dimensional measure of the boundary of A (i.e., the *perimeter* of A). The overall behaviour of such energy for large sets (compared with the lattice spacing) can be described by scaling it as

$$E_\varepsilon(A) = \varepsilon^{d-1}\#\big\{(i, j)\colon\ i \in A,\ j \notin A,\ |i - j| = \varepsilon\big\} \quad \text{for } A \subset \varepsilon\mathbb{Z}^d,$$