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Homomorphisms of convolution algebras

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Abstract

We establish an explicit, algebraic, one-to-one correspondence between the $*$-homomorphisms, $\varphi : L^1(F) \to M(G)$, of group and measure algebras over locally compact groups F and G, and group homomorphisms, $\phi : F \to \mathbb{M}_\phi$, where $\mathbb{M}_\phi$ is a semi-topological subgroup of $(M(G), w^*)$. We show how to extend any such $*$-homomorphism to a larger convolution algebra to obtain nicer continuity properties. We augment Greenleaf's characterization of the contractive subgroups of $M(G)$ (Greenleaf, 1965 [17]) by completing the description of their topological structures. We show that not every contractive homomorphism has the dual form of Cohen's factorization in the abelian case, thus answering a question posed by Kerlin and Pepe (1975) in [27]. We obtain an alternative factorization of any contractive homomorphism $\varphi : L^1(F) \to M(G)$ into four homomorphisms, where each of the four factors is one of the natural types appearing in the Cohen factorization.

Keywords: Locally compact group; Group algebra; Measure algebra; Homomorphism; Introverted subspace

Let F and G be locally compact groups. The convolution "homomorphism problem" in abstract harmonic analysis asks for a description of all bounded homomorphisms between group and measure algebras, $L^1(F)$ and $M(G)$ (and related convolution algebras); the dual version of the homomorphism problem asks for a description of all homomorphisms between Fourier and Fourier–Stieltjes algebras, $A(F)$ and $B(G)$. Wendel's theorem, which describes the isometric isomorphisms between group algebras $L^1(F)$ and $L^1(G)$ [40], is among the first significant contributions towards a solution to this old problem. When F and G are

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abelian, Paul Cohen solved (both versions of) the homomorphism problem in 1960 [2,34]. Since the publication of [2], many mathematicians have worked towards extending Cohen's beautiful theorem to the nonabelian setting and in 1965, Greenleaf [17] successfully provided a characterization all contractive homomorphisms $\varphi : L^1(F) \to M(G)$ for arbitrary groups. As noted by Kerlin and Pepe [27], this characterization is less tractable than the one obtained by Cohen in the abelian case. Thus, beyond the general convolution homomorphism problem, an open problem has asked for a factorization of contractive homomorphisms that more closely shares the spirit of Cohen's theorem. Over the years this last question has been answered when either F or G is assumed to be abelian, some simplifications to Greenleaf's original arguments have been obtained, and (isometric) isomorphisms of convolution algebras have been intensely investigated – for example, see [25,38,27,26,31,13,14, 41,15,16,11,5]. However, since the publication of [17], very little progress has been made with the general version of the convolution homomorphism problem.

In the dual situation, isometric isomorphisms of Fourier and (reduced) Fourier–Stieltjes algebras are characterized in [39] and [29]. (The isometric isomorphism theorems of Wendel, Johnson, Strichartz, and Walter are extended to the situation of Kac algebras in [6]; also see [7].) M. Ilie and N. Spronk employed the operator space structures of $A(F)$ and $B(G)$ to extend Cohen's theorem, and an intermediate result due to Host [20], to the dual setting when F is amenable [21] and [22]. The dual version of the homomorphism problem – which in general also remains open – has since enjoyed a revived period of investigation; for example, see [23,24,33]. In particular, H.L. Pham recently showed that every contractive homomorphism $\varphi : A(F) \to B(G)$ factors as $\varphi = l_{r_0} \circ s \circ j_\theta \circ l_{u_0}$ where $l_r u(s) = u(rs)$; $\theta : G_0 \to F$ is a continuous homomorphism, or anti-homomorphism, defined on an open subgroup G_0 of G and $j_\theta u = u \circ \theta$; and $s : B(G_0) \to B(G)$ is the expansion homomorphism given by $s(u)(h) = u(h)$ if $h \in G_0$, $s(u)(h) = 0$ otherwise [33, Theorem 5.1]. This extends Cohen's characterization of all homomorphisms $\varphi : A(F) \to B(G)$ in the abelian setting and is also consistent with Ilie's and Spronk's work.

$$\begin{array}{ccccc} A(F) & & \xrightarrow{\quad\varphi\quad} & & B(G) \\ {\scriptstyle l_{u_0}}\downarrow & & & & \uparrow{\scriptstyle l_{r_0}} \\ A(F) & \xrightarrow{j_\theta} & B(G_0) & \xrightarrow{s} & B(G) \end{array} \qquad \begin{array}{ccccc} L^1(F) & & \xrightarrow{\quad\varphi\quad} & & M(G) \\ {\scriptstyle A_\alpha}\downarrow & & & & \uparrow{\scriptstyle A_{\rho_0}} \\ L^1(F) & \xrightarrow{j^*_{\theta_K}} & M(G/K) & \xrightarrow{S^*_K} & M(G) \end{array} \tag{0.1}$$

Assuming that F and G are abelian, one can check that the precise dual form of the homomorphisms l_r, j_θ and s are the maps $A_\alpha = M^*_\alpha$, $j^*_{\theta_K}$ and S^*_K – defined precisely in Section 5 – where α is a continuous character, $\theta_K : F \to G/K$ is a continuous homomorphism and K is a compact subgroup of G. An open problem, posed explicitly in [27], thus asks whether every contractive homomorphism $\varphi : L^1(F) \to M(G)$ has the factorization illustrated in the second of the above diagrams when F and G are not assumed to be abelian. Kerlin and Pepe answered this question affirmatively in the case when G is abelian (and a different proof of this fact is given in Section 5). With Example 5.4 we will give a negative answer to this question, in general. Our main result of Section 5, Theorem 5.11, provides an alternative factorization of any contractive homomorphism φ into four, canonically defined, homomorphisms, $\varphi = j^*_{\theta_H} \circ A_{\alpha_{\mathbb{T}}} \circ S^*_{\Omega_\rho} \circ j^*_\theta$. We note that the group and measure algebra homomorphisms appearing in our factorization are of