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Approximation properties and Schauder decompositions in Lipschitz-free spaces ☆

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Abstract

We prove that the Lipschitz-free space over a doubling metric space has the bounded approximation property. We also show that the Lipschitz-free spaces over ℓ_1^N or ℓ_1 have monotone finite-dimensional Schauder decompositions.

Keywords: Lipschitz-free spaces; Approximation property; Schauder decompositions

1. Introduction

For (M_1, d_1) and (M_2, d_2) metric spaces and $f : M_1 \to M_2$, we denote by $\mathrm{Lip}(f)$ the Lipschitz constant of f given by

$$\mathrm{Lip}(f) = \sup\left\{\frac{d_2(f(x), f(y))}{d_1(x, y)},\ x, y \in M_1,\ x \neq y\right\}.$$

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Consider (M,d) a *pointed* metric space, i.e. a metric space equipped with a distinguished element (origin) denoted 0. Then, the space $\mathrm{Lip}_0(M)$ of all real-valued Lipschitz functions f on M which satisfy $f(0)=0$, endowed with the norm

$$\|f\|_{\mathrm{Lip}_0(M)} = \mathrm{Lip}(f)$$

is a Banach space. The Dirac map $\delta : M \to \mathrm{Lip}_0(M)^*$ defined by $\langle g, \delta(p)\rangle = g(p)$ for $g \in \mathrm{Lip}_0(M)$ and $p \in M$ is an isometric embedding from M into $\mathrm{Lip}_0(M)^*$. The closed linear span of $\{\delta(p),\ p \in M\}$ is denoted $\mathcal{F}(M)$ and called the *Lipschitz-free space over* M (or free space in short). It follows from the compactness of the unit ball of $\mathrm{Lip}_0(M)$ with respect to the topology of pointwise convergence, that $\mathcal{F}(M)$ can be seen as the canonical predual of $\mathrm{Lip}_0(M)$. Then the weak*-topology induced by $\mathcal{F}(M)$ on $\mathrm{Lip}_0(M)$ coincides with the topology of pointwise convergence on the bounded subsets of $\mathrm{Lip}_0(M)$. Lipschitz-free spaces are a very useful tool for abstractly linearizing Lipschitz maps. Indeed, if we identify through the Dirac map a metric space M with a subset of $\mathcal{F}(M)$, then any Lipschitz map from the metric space M to a metric space N extends to a continuous linear map from $\mathcal{F}(M)$ to $\mathcal{F}(N)$ with the same Lipschitz constant (see [14] or Lemma 2.2 in [5]). A comprehensive reference for the basic theory of the spaces of Lipschitz functions and their preduals, which are called Arens–Eells spaces there, is the book [14] by Weaver.

Despite the simplicity of their definition, very little is known about the linear structure of Lipschitz-free spaces over separable metric spaces. It is easy to see that $\mathcal{F}(\mathbb{R})$ is isometric to L_1. However, adapting a theorem of Kislyakov [8], Naor and Schechtman proved in [12] that $\mathcal{F}(\mathbb{R}^2)$ is not isomorphic to any subspace of L_1. Then the metric spaces whose Lipschitz-free space is isometric to a subspace of L_1 have been characterized by Godard in [4].

The aim of this paper is to study metric spaces M such that $\mathcal{F}(M)$ has the bounded approximation property (BAP) or admits a finite-dimensional Schauder decomposition (FDD). This kind of study was initiated in the fundamental paper by Godefroy and Kalton [5], where they proved that a Banach space X has the λ-BAP if and only if $\mathcal{F}(X)$ has the λ-BAP. In particular, for any finite-dimensional Banach space E, $\mathcal{F}(E)$ has the metric approximation property (MAP). Another major result from [5] is that any separable Banach space has the so-called isometric lifting property. Refining the techniques used in the proof of this result, Godefroy and Ozawa have proved in their recent work [6] that any separable Banach space failing the BAP contains a compact subset whose Lipschitz-free space also fails the BAP. It is then natural, as it is suggested in [6], to try to describe the metric spaces whose Lipschitz-free space has BAP. We address this question in Section 2. Our main result of this section (Corollary 2.2) is that for any doubling metric space M, the Lipschitz-free space $\mathcal{F}(M)$ has the BAP.

Then we try to find the Banach spaces such that the corresponding Lipschitz-free spaces have stronger approximation properties. The first result in this direction is due to Borel-Mathurin [1], who proved that $\mathcal{F}(\mathbb{R}^N)$ admits a finite-dimensional Schauder decomposition. The decomposition constant obtained in [1] depends on the dimension N. In Section 3 we show that $\mathcal{F}(\ell_1^N)$ and $\mathcal{F}(\ell_1)$ admit a monotone finite-dimensional Schauder decomposition. For that purpose, we use a particular technique for interpolating Lipschitz functions on hypercubes of $\mathbb{R}^N$.

2. Bounded approximation property for Lipschitz-free spaces and gentle partitions of unity

We first recall the definition of the bounded approximation property.