Available online at www.sciencedirect.com
SciVerse ScienceDirect
Journal of Functional Analysis 264 (2013) 2356–2385
JOURNAL OF Functional Analysis
www.elsevier.com/locate/jfa

Estimates for rough Fourier integral and pseudodifferential operators and applications to the boundedness of multilinear operators ☆

Salvador Rodríguez-López *, Wolfgang Staubach

Department of Mathematics, Uppsala University, Uppsala, SE 75106, Sweden

Received 18 October 2012; accepted 23 February 2013
Available online 5 March 2013
Communicated by I. Rodnianski

Abstract

We study the boundedness of rough Fourier integral and pseudodifferential operators, defined by general rough Hörmander class amplitudes, on Banach and quasi-Banach L^p spaces. Thereafter we apply the aforementioned boundedness in order to improve on some of the existing boundedness results for Hörmander class bilinear pseudodifferential operators and certain classes of bilinear (as well as multilinear) Fourier integral operators. For these classes of amplitudes, the boundedness of the aforementioned Fourier integral operators turn out to be sharp. Furthermore we also obtain results for rough multilinear operators.

Keywords: Fourier integral operators; Pseudodifferential operators; Multilinear operators

1. Introduction and summary of the results

A (linear) Fourier integral operator or FIO for short, is an operator that can be written locally in the form

☆ Both authors were supported by the EPSRC First Grant Scheme reference number EP/H051368/1. The first author is also partially supported by the grant MTM2010-14946.
* Corresponding author.
E-mail addresses: salvador@math.uu.se (S. Rodríguez-López), wulf@math.uu.se (W. Staubach).

0022-1236/$ – see front matter
http://dx.doi.org/10.1016/j.jfa.2013.02.018

$$T_a f(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i\varphi(x,\xi)} a(x,\xi) \hat{f}(\xi)\, \mathrm{d}\xi,$$

where $a(x,\xi)$ is the *amplitude*, $\varphi(x,\xi)$ is the *phase function* and f belongs to $C_0^\infty(\mathbb{R}^n)$. In case the phase function $\varphi(x,\xi) = \langle x,\xi\rangle$, the Fourier integral operator is called a pseudodifferential operator, which in what follows will be abbreviated as ΨDO. The study of these operators, which are intimately connected to the theory of linear partial differential operators, has a long history. There is a large body of results concerning the regularity, e.g. the L^p boundedness, of FIOs and ΨDOs, but due to the lack of space we only mention those investigations that are of direct relevance to the current paper.

The most widely used class of amplitudes are those introduced by Hörmander in [16], the so-called $S^m_{\rho,\delta}$ class, that consists of $a(x,\xi) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ with

$$\left|\partial_\xi^\alpha \partial_\xi^\beta a(x,\xi)\right| \leqslant C_{\alpha\beta}\big(1+|\xi|\big)^{m-\rho|\alpha|+\delta|\beta|},$$

$m \in \mathbb{R}$, $\rho,\delta \in [0,1]$. For phase functions one usually assumes that $\varphi \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n \setminus 0)$ is homogeneous of degree 1 in the frequency variable ξ and satisfies the *non-degeneracy condition*, that is the mixed Hessian matrix $[\frac{\partial^2\varphi}{\partial x_j \partial \xi_k}]$ has non-vanishing determinant.

The sharp local L^p ($p \in (1,\infty)$) boundedness of FIOs, under the assumptions of $a(x,\xi) \in S^m_{\rho,1-\rho}$ being compactly supported in the spatial variable x and $\rho \in [\frac{1}{2},1]$, $m = (\rho-n)|1/p - 1/2|$ was established by A. Seeger, C.D. Sogge and E.M. Stein [22]. The global L^p boundedness of FIOs (i.e. boundedness without the assumption of compact spatial support of the amplitude) has also been investigated in various contexts and here we would like to mention boundedness of operators with smooth amplitudes in the so-called **SG** classes, due to E. Cordero, F. Nicola and L. Rodino in [8]; the boundedness of operators with amplitudes in $S^m_{1,0}$ on the space of compactly supported distributions whose Fourier transform is in $L^p(\mathbb{R}^n)$ (i.e. the $\mathscr{F}L^p$ spaces) due to Cordero, Nicola and Rodino in [7] and Nicola's refinement of this investigation in [20]; and finally, S. Coriasco and M. Ruzhansky's global L^p boundedness of Fourier integral operators [9], with amplitudes that belong to a certain subclass of $S^0_{1,0}$.

In this paper we consider the problem of boundedness of Fourier integral operators with amplitudes that are non-smooth in the spatial variables and exhibit an L^p type behavior in those variables for $p \in [1,\infty]$. This is a continuation of the investigation of boundedness of rough pseudodifferential operators made by C. Kenig and W. Staubach [18] and that of regularity of rough Fourier integral operators carried out by D. Dos Santos Ferreira and W. Staubach [10], where the boundedness of the aforementioned operators were established under the condition that the corresponding amplitudes are L^∞ functions in the spatial variables.

One motivation for the study of these specific classes of rough oscillatory integrals whose amplitudes have L^p spatial behavior is, as will be demonstrated in this paper, its applicability in proving boundedness results for multilinear pseudodifferential and Fourier integral operators.

A study of rough pseudodifferential operators without any regularity assumption in the spatial variables were carried out in [18] and A. Stefanov's paper [23], where the irregularity of the symbols of the operators are of L^∞ type. Prior to these investigations, a systematic study of pseudodifferential operators with limited smoothness was carried out by M. Taylor in [25]. The corresponding problem for the Fourier integral operators were investigated in [10]. In the case of L^p spatial behavior, an investigation of boundedness of pseudodifferential operators was carried out by N. Michalowski, D. Rule and W. Staubach in [19]. To our knowledge, there has been no