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# Positive operators and maximal operators in a filtered measure space

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**Abstract**

In a filtered measure space, a characterization of weights for which the trace inequality of a positive operator holds is given by the use of discrete Wolff's potential. A refinement of the Carleson embedding theorem is also introduced. Sawyer type characterization of weights for which a two-weight norm inequality for a generalized Doob's maximal operator holds is established by an application of our Carleson embedding theorem. Moreover, Hytönen–Pérez type one-weight norm estimate for Doob's maximal operator is obtained by the use of our two-weight characterization.


*Keywords:* $A_p$ weight; Conditional expectation; Positive operator; Discrete Wolff's potential; Filtered measure space; Martingale; Two-weight norm inequality; Sawyer type condition; The Carleson embedding theorem; Trace inequality

## 1. Introduction

Weighted Norm Inequalities in Harmonic analysis is an old subject whose systematic investigation was initiated by [38,8,39] etc. A classical reference in the field is [11].

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Dyadic Harmonic Analysis has recently acquired a renewed attention because of its wide applicability to Classical Harmonic Analysis, including weighted norm inequalities. Petermichl [43] and Nazarov, Treil and Volberg [41] were cornerstone works, whose investigations have been continued by many authors. This subject is also old, which can be found in [44] and [12] etc. For more complete references, we refer to the bibliographies of [41,31].

Two of the important topics in the intersection of these subjects are to get sharp one-weight estimates of usual operators in Classical Harmonic Analysis and to get necessary and sufficient conditions of weights for the boundedness of those operators in the two-weight setting. Interestingly, these two topics are closely related. One way to attack these problems is a dyadic discretization technique. For the first problem, one of the important steps of a solution is getting a sharp one-weight estimate for a dyadic discretization of a singular integral operator, i.e., a generalized Haar shift operator. A sharp one-weight estimate of general singular integral operators, i.e., the $A_2$-conjecture, which has been an open problem in this field for a long time, was settled by Hytönen [18] along this line and its simpler proofs were found by several authors (cf. [23,35] etc.). For (linear) positive operators, one example of which is a fractional integral operator, investigations along this line were done by several authors [30,46,47,54,4,5] and more recently by [33,32,27,26,52]. For the Hardy–Littlewood maximal operator (including a fractional maximal operator), Sawyer [44] got a two-weight characterization by considering the dyadic Hardy–Littlewood (fractional) maximal operator. Recently, using similar techniques, the sharp weighted estimates of the Hardy–Littlewood (fractional) maximal operator are established in the works [34,33,19,21], which are continuations of the work of Buckley [3]. For a survey of these developments, we refer to [42,17,16].

On the other hand, Martingale Harmonic Analysis is a subject which has also been well studied. Doob's maximal operator, which is a generalization of the dyadic Hardy–Littlewood maximal operator, and a martingale transform, which is an analogue of a singular integral in Classical Harmonic Analysis, are important tools in stochastic analysis. This field is called Martingale Harmonic Analysis and is well explained in the books by Dellancherie and Meyer [10], Long [36] and Kazamaki [28]. For Doob's maximal operator, one-weight estimate was studied first by Izumisawa and Kazamaki [24], assuming some regularity condition on $A_p$ weights. Later, Jawerth [25] found that the added property is superfluous (see Remark 4.6 below). For two-weight norm inequalities, the first study is done by Uchiyama [53], concerning necessary and sufficient condition of weights for weak type $(p, p)$ inequalities to hold. Concerning strong $(p, q)$ type inequalities, Long and Peng [37] found necessary and sufficient conditions for weights, which is the analogous to Sawyer's condition for the boundedness of the Hardy–Littlewood maximal operator. There is also a recent work by Chen and Liu [7] on this topic. For positive operators, there seems no work done in a filtered probability space or in a filtered measure space and we shall try to generalize the results in the Euclidean space of the weighted estimate for dyadic positive operators to those in a martingale setting. (For fractional integral operators in a martingale setting, there is a recent work by Nakai and Sadasue [40].)

The study of a boundedness property of positive operators and maximal operators is closely related to the Carleson embedding (or measure) theorem, which is a martingale analogue of the Carleson embedding theorem of a Hardy space into a weighted Lebesgue space. In the dyadic setting in the Euclidean space, this coincides with the Dyadic Carleson embedding theorem. The Carleson measure in a continuously filtered probability space was first introduced by Arai [1] with an application to the corona theorem on Complex Brownian Spaces. This was rediscovered later by Long [36] in a discrete case, with an application to a characterization of *BMO* martingales.