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A quantitative isoperimetric inequality for fractional perimeters

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Abstract

Recently Frank and Seiringer have shown an isoperimetric inequality for nonlocal perimeter functionals arising from Sobolev seminorms of fractional order. This isoperimetric inequality is improved here in a quantitative form.

Keywords: Quantitative isoperimetric inequality; Fractional perimeter; Fractional Sobolev spaces

1. Introduction

Isoperimetric inequalities play a crucial role in many areas of mathematics such as geometry, linear and nonlinear PDEs, or probability theory. In the Euclidean setting, it states that among all sets of prescribed measure, balls have the least perimeter. More precisely, for any Borel set $E \subset \mathbb{R}^N$ of finite Lebesgue measure,

$$N|B|^{1/N}|E|^{(N-1)/N} \leqslant P(E), \tag{1.1}$$

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where B denotes the unit ball of $\mathbb{R}^N$ centered at the origin. Here $P(E)$ denotes the distributional perimeter of E which coincides with the $(N-1)$-dimensional measure of ∂E when E has a (piecewise) smooth boundary. It is a well-known fact that inequality (1.1) is strict unless E is a ball. Here the natural framework for studying the isoperimetric inequality is the theory of sets of finite perimeter. We briefly recall that a Borel set E of finite Lebesgue measure is said to be of finite perimeter if its characteristic function χ_E belongs to $BV(\mathbb{R}^N)$, and then $P(E)$ is given by the total variation of the distributional derivative of χ_E. Throughout this paper, we shall refer to the monograph [4] for the basic properties of sets of finite perimeter.

The *isoperimetric deficit* of a set E of finite perimeter is defined as the scaling and translation invariant quantity

$$D(E) := \frac{P(E) - P(B_r)}{P(B_r)},$$

where $B_r := rB$ is the ball having the same measure as E, *i.e.*, $r^N|B| = |E|$. By the characterization of the equality cases in (1.1), the isoperimetric inequality rewrites $D(E) \geqslant 0$, and $D(E) = 0$ if and only if E is a translation of B_r. Hence the isoperimetric deficit measures in some sense how far is a set from being ball. Finding a quantitative version of (1.1) consists in proving that the isoperimetric deficit controls a more usual notion of "distance from the family of the balls". To this aim is introduced the so-called *Fraenkel asymmetry* of the set E, and it is defined by

$$A(E) := \min\left\{\frac{|E \triangle B_r(x)|}{|E|}\colon x \in \mathbb{R}^N,\ r^N|B| = |E|\right\},$$

where $B_r(x) := x + rB$, and $\triangle$ denotes the symmetric difference between sets. Note that asymmetry is also invariant under scaling and translations. We then look for a positive constant C_N depending only on the dimension, and an exponent $\alpha > 0$ such that $A(E) \leqslant C_N(D(E))^\alpha$, which can be rewritten as a quantitative form of (1.1),

$$P(E) \geqslant \left(1 + \left(\frac{A(E)}{C_N}\right)^{1/\alpha}\right) N|B|^{1/N}|E|^{(N-1)/N}.$$

We shall not attempt here to sketch the history of this problem, but simply refer to the recent paper by Fusco, Maggi and Pratelli [17] (and references therein) where this inequality has been first proved with the optimal exponent $\alpha = 1/2$, and to Figalli, Maggi and Pratelli [14] for anisotropic perimeter functionals (see also [12], and [19] for a survey).

The main goal of this paper is to prove a quantitative isoperimetric type inequality for nonlocal perimeter functionals arising from Sobolev seminorms of fractional order. First, let us introduce what we call the fractional s-perimeter of a set. For $s \in (0, 1)$ and a Borel set $E \subset \mathbb{R}^N$, $N \geqslant 1$, we define the fractional s-perimeter of E by

$$P_s(E) := \int_E \int_{E^c} \frac{1}{|x-y|^{N+s}}\, dx\, dy.$$

If $P_s(E) < \infty$, we observe that

$$P_s(E) = \frac{1}{2}[\chi_E]^p_{W^{\sigma,p}(\mathbb{R}^N)}, \tag{1.2}$$