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Journal of Functional Analysis 261 (2011) 1145–1203

JOURNAL OF Functional Analysis

www.elsevier.com/locate/jfa

On the hereditary proximity to ℓ_1

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Received 24 July 2009; accepted 23 April 2011

Available online 17 May 2011

Communicated by K. Ball

Abstract

In the first part of the paper we present and discuss concepts of local and asymptotic hereditary proximity to ℓ_1. The second part is devoted to a complete separation of the hereditary local proximity to ℓ_1 from the asymptotic one. More precisely for every countable ordinal ξ we construct a separable Hereditarily Indecomposable reflexive space $\mathfrak{X}_\xi$ such that every infinite-dimensional subspace of it has Bourgain ℓ_1-index greater than ω^ξ and the space itself has no ℓ_1-spreading model.

Keywords: Bourgain ℓ_1-index; ℓ_1-Spreading model; Attractors method

1. Introduction

Concepts of proximity to a classical ℓ_p (or c_0) space play a significant role to the understanding of the structure of a Banach space. They are categorized as follows:

The first one is the global proximity to ℓ_p which simply means that ℓ_p is isomorphic to a subspace Y of X. The local proximity which occurs more frequently, due to J.L. Krivine's theorem [21], is measured through the Bourgain ℓ_p-index [10]. The last concept is the asymptotic proximity that varies from A. Brunel and L. Sucheston ℓ_p-spreading models [11], to

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1 Research partially supported by *ΠΕΒΕ* 2009 NTUA Research Program.

0022-1236/$ – see front matter
doi:10.1016/j.jfa.2011.04.012

the asymptotic ℓ_p spaces. The latter class of Banach spaces, introduced by V. Milman and N. Tomczak-Jaegermann in [25], is modelled on B.S. Tsirelson space [31] that answered in negative the famous Banach's problem by showing that global proximity to some ℓ_p is not always possible.

It is well known that the hereditary global proximity, i.e. saturation by subspaces isomorphic to ℓ_p, does not imply asymptotic structure of the whole space, see for example [19,18]. If we allow passing to infinite-dimensional subspaces, it is easy to see that the global proximity to ℓ_p is the strongest one followed by the asymptotic one. The local proximity is the weakest among them. It is also known that the three classes are separated for each ℓ_p. Namely there are spaces with arbitrarily large local proximity to ℓ_p but admitting no ℓ_p asymptotic subspace, and similarly there are ℓ_p asymptotic spaces with no subspaces isomorphic to ℓ_p. The present paper is mainly devoted to the separation of the local and asymptotic proximity to ℓ_1 when the first one is hereditarily large. In particular our work is motivated by a result of the third named author stated as follows.

Theorem. *(See [28].) Let X be a separable Banach space and ξ be a countable ordinal. If X is boundedly distortable and has hereditary Bourgain ℓ_1-index greater than ω^ξ then it is saturated by asymptotic ℓ_1^ξ spaces.*

Let's recall that the hereditary Bourgain ℓ_p-index of a Banach space X is the minimum of Bourgain ℓ_p-index of its subspaces. In the sequel by the ℓ_p-index we will mean the Bourgain ℓ_p-index.

In view of the above theorem it is natural to ask how critical is the bounded distortion of X for the final conclusion. It is also worth adding that heredity assumptions for the local proximity to ℓ_1 could yield large asymptotic one. In this direction we prove the following

Proposition. *Let (e_n) be a Schauder basis of a Banach space X such that the Bourgain ℓ_1-tree supported by any subsequence of $(e_n)_{n\in\mathbb{N}}$ has order greater than ω^ξ. Then there exists a subsequence generating an ℓ_1^ξ-spreading model.*

Our aim is to show that large hereditary ℓ_1-structure in a Banach space X does not imply in general any asymptotic one. More precisely the main goal at the present paper is to prove the next

Theorem A. *For every countable ordinal ξ there exists a separable Hereditarily Indecomposable reflexive space $\mathfrak{X}_\xi$ with the hereditary ℓ_1-index greater than ω^ξ such that $\mathfrak{X}_\xi$ does not admit an ℓ_1-spreading model. Moreover the dual $\mathfrak{X}_\xi^*$ has hereditary c_0-index greater than ω^ξ and does not admit c_0 as a spreading model.*

The space $\mathfrak{X}_\xi$ satisfies the following structural property yielding that its hereditary ℓ_1 index is greater than ω^ξ. For any block sequence $(x_n)_n$ of the basis $(e_n)_n$ of $\mathfrak{X}_\xi$ there exists a further block $(x_s)_{s\in\mathcal{T}}$ with $\mathcal{T}$ a well-founded tree of order greater than ω^ξ such that for each chain $s_1,\dots,s_k$ in $\mathcal{T}$, $(x_{s_i})_{i=1}^k$ is C-equivalent to the standard basis $(e_i)_{i=1}^k$ of ℓ_1^k for a universal constant C.

To some extent the spaces $\mathfrak{X}_\xi$, $\xi<\omega_1$, are the reflexive analogue of the famous Gowers tree-space (cf. [15]) and its variants (cf. [3]). For the construction of $\mathfrak{X}_\xi$ we employ the method of attractors, appeared in [8] and extensively used in [3]. The definition of the space $\mathfrak{X}_\xi$ requires