Available online at www.sciencedirect.com
ScienceDirect
Journal of Functional Analysis 261 (2011) 1345–1360
JOURNAL OF Functional Analysis
www.elsevier.com/locate/jfa

A semi-finite algebra associated to a subfactor planar algebra

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Received 4 May 2011; accepted 6 May 2011
Available online 18 May 2011
Communicated by Alain Connes

Abstract

We canonically associate to any planar algebra two type II_∞ factors $\mathfrak{M}_\pm$. The subfactors constructed previously by the authors in Guionnet et al. (2010) [6] are isomorphic to compressions of $\mathfrak{M}_\pm$ to finite projections. We show that each $\mathfrak{M}_\pm$ is isomorphic to an amalgamated free product of type I von Neumann algebras with amalgamation over a fixed discrete type I von Neumann subalgebra. In the finite-depth case, existing results in the literature imply that $\mathfrak{M}_+ \cong \mathfrak{M}_-$ is the amplification a free group factor on a finite number of generators. As an application, we show that the factors M_j constructed in Guionnet et al. (in press) [6] are isomorphic to interpolated free group factors $L(\mathbb{F}(r_j))$, $r_j = 1 + 2\delta^{-2j}(\delta - 1)I$, where δ^2 is the index of the planar algebra and I is its global index. Other applications include computations of laws of Jones–Wenzl projections.

Keywords: Von Neumann algebras; Free probability; Subfactors

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[1] Research supported by ANR project ANR-08-BLAN-0311-01.
[2] Research supported by NSF grant DMS-0856316.
[3] Research supported by NSF grants DMS-0555680, DMS-0900776.

0022-1236/$ – see front matter
doi:10.1016/j.jfa.2011.05.004

1. Introduction

In this paper, we associate a pair of semi-finite von Neumann algebras $\mathfrak{M}_\pm$ to a planar algebra $\mathcal{P}$. The algebras $\mathfrak{M}_\pm$ are obtained via the GNS construction from a certain non-unital tracial inductive limit algebra V_+ which arises canonically from $\mathcal{P}$. These algebras have an interesting structure, and the paper is mainly devoted to their study.

To state our main application, let $\mathcal{P}$ be a subfactor planar algebra of index δ^2, and let us denote by $M_k = M_k(\mathcal{P})$ the von Neumann algebra generated in the GNS representation of $(\mathcal{P}, \wedge_k, Tr_k)$ (see Def. 7 and 8 in [6]). We prove:

Theorem 1. *Assume that $\mathcal{P}$ is finite-depth with global index I. Then $M_k \cong L(\mathbb{F}(r_k))$ with $r_k = 1 + 2\delta^{-2k}(\delta - 1)I$.*

We refer the reader to [9,5] for the definition of global index I. If Γ is the principal graph of $\mathcal{P}$ and μ is the Perron–Frobenius eigenvector normalized by $\mu(*) = 1$, then $I = \frac{1}{2}\sum_{v\in\Gamma}\mu(v)^2$. This formula is consistent with the result of Kodiyalam and Sunder in the depth two case [7].

The main step in proving Theorem 1 is to prove that the amplifications of M_k are isomorphic to type II_∞ von Neumann algebra $\mathfrak{M}_+$ or $\mathfrak{M}_-$ (the choice of sign is according to the parity of k). It turns out that each $\mathfrak{M}_\pm$ admits a description as a (possibly infinite) free product with amalgamation over a discrete type I von Neumann subalgebra of type I von Neumann algebras. In the finite-depth case, this is sufficient to determine the isomorphism class of $\mathfrak{M}_\pm$ using the work of Dykema [1–4].

We note that r_k in the statement of Theorem 1 satisfy $(r_k - 1) = \delta^2(r_{k+1} - 1)$.

We mention also that while this paper was in preparation, Kodiyalam and Sunder have found a different proof that M_k are (in the finite-depth case) isomorphic to interpolated free group factors [8].

We conclude the paper with another application of the isomorphism between M_0 and a compression of $\mathfrak{M}$. It allows us to recover the random matrix model used in [6] and can be quite useful in random matrix computations (we illustrate this by describing the joint law of Jones–Wenzl idempotents JW).

2. A semi-finite tracial algebra associated to a planar algebra

Let $\mathcal{P}$ be a planar algebra (which will be shortly assumed to be a subfactor planar algebra). We denote by $\mathcal{P}_k^\epsilon$, $k = 0, 1, 2, \ldots$, $\epsilon = \pm$, the k-th graded component of $\mathcal{P}$.

For fixed k, ϵ and integers a, b, p satisfying $a + b + p = 2k$, let $V_{a,b}^\epsilon(p)$ be a copy of $\mathcal{P}_k^\epsilon$; thus by convention tangles acting on elements of $V_{a,b}^\epsilon(p)$ are drawn with input disks to have, clockwise from the first string, a strings on the left, p on top, b on the right and so that the top-left corner has shading ϵ. Define the multiplication map

$$\cdot : V_{a,b}^\epsilon(p) \times V_{a',b'}^{\epsilon'}(p') \to V_{a,b'}^\epsilon(p + p')$$

to be zero unless $b = a'$ and $\epsilon' = (-1)^p\epsilon$ and otherwise by the tangle: