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ScienceDirect
Journal of Functional Analysis 260 (2011) 340–398
JOURNAL OF Functional Analysis
www.elsevier.com/locate/jfa

Perturbations of embedded eigenvalues for the planar bilaplacian

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Received 11 September 2009; accepted 3 October 2010

Communicated by I. Rodnianski

Abstract

Operators on unbounded domains may acquire eigenvalues that are embedded in the essential spectrum. Determining the fate of these embedded eigenvalues under small perturbations of the underlying operator is a challenging task, and the persistence properties of such eigenvalues are linked intimately to the multiplicity of the essential spectrum. In this paper, we consider the planar bilaplacian with potential and show that the set of potentials for which an embedded eigenvalue persists is locally an infinite-dimensional manifold with infinite codimension in an appropriate space of potentials.

Keywords: Embedded eigenvalues; Persistence; Perturbation; Bilaplacian

1. Introduction

Determining the dependence of the spectrum of operators on perturbations is an important issue that is of relevance in many applications. Of course, much is known in this direction: the persistence of point eigenvalues and the behaviour of the essential spectrum under small bounded

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[1] Partially supported by the European Union under the Marie Curie Fellowship MEIF-CT-2005-024191 and by the Swedish Research Council.
[2] Partially supported by a Royal Society Wolfson Research Merit Award and by the NSF under grant DMS-0907904.

0022-1236/$ – see front matter
doi:10.1016/j.jfa.2010.10.001

perturbations, for instance, have been analysed comprehensively, and we refer to [11] for many results along these lines. Here, we consider differential operators that are posed on unbounded domains and are interested in the interaction between eigenvalues, with proper eigenfunctions in the underlying domain of the operator, and the essential spectrum. More precisely, we study the fate of eigenvalues that are embedded in the essential spectrum under small perturbations of the operator. Typically, such eigenvalues will disappear under generic perturbations of the potential, and it is therefore of interest to determine the class of perturbations for which an embedded eigenvalue persists. For the bilaplacian on cylindrical domains, we showed in our previous work [8] that the set of perturbations for which an embedded eigenvalue persists is an infinite-dimensional manifold of finite codimension. Furthermore, we showed that the codimension of this set is given by the multiplicity of the essential spectrum, defined as the number of independent continuum eigenfunctions or, more rigorously, via the spectral resolution of the Fourier transform of the bilaplacian (see e.g. [3, Definition 2 in §85]). In this paper, we continue the investigation that we began in [8] and consider the bilaplacian posed on the plane: the challenge is that the essential spectrum of the planar bilaplacian has infinite multiplicity. Thus, we may expect that the set of potentials for which an embedded eigenvalue persists is an infinite-dimensional manifold of infinite codimension, and this is indeed what we shall prove for an appropriate class of potentials. For a different approach on persistence of embedded eigenvalues, see [2].

Before stating our results, we briefly outline why embedded eigenvalues are of interest. Our first motivation comes from quantum mechanics: the eigenfunctions associated with eigenvalues of an energy operator correspond to bound states that can be attained by the physical system modelled by the energy operator. If such an eigenvalue is embedded in the essential spectrum, then its fate under perturbations of the potential determines whether the associated bound states persist or not (see [10,16] for examples). The second example comes from inverse scattering theory, where eigenvalues correspond to coherent soliton structures of the underlying integrable system, while the essential spectrum describes radiative scattering behaviour. Thus, bifurcations of solitons are reflected by the disappearance or persistence of embedded eigenvalues [13,14]. Finally, embedded eigenvalues provide a common mechanism for the destabilisation of travelling waves in near-integrable Hamiltonian partial differential equations, and we refer to [17] for further background information and pointers to the literature.

As mentioned above, we focus in this paper on the persistence of embedded eigenvalues for the planar bilaplacian. Our primary reason for considering the bilaplacian is that this operator is complex enough to exhibit the underlying difficulties, while not adding technical complications that have nothing to do with the issue we are interested in. In other words, the planar bilaplacian provides a useful paradigm for the issues that we expect to encounter for other more complicated differential operators. Note also that the applications we mentioned above all involve self-adjoint operators, a feature shared by the bilaplacian.

We now describe the precise setting that we consider. Let $r_0 > 0$, and assume that $\theta \in C_0^\infty(B_{r_0}(0); \mathbb{R})$ is a radially symmetric potential. Hence, we use polar coordinates (r, φ), write $\theta = \theta(r)$, and consider the multiplication operator on $L^2(\mathbb{R}^2)$ (also denoted by θ) defined by

$$[\theta u](r, \varphi) := \theta(r) u(r, \varphi).$$

We define $\mathcal{L} := \Delta^2 + \theta$ on $L^2(\mathbb{R}^2)$, where Δ^2 is the bilaplace operator which is densely defined on $L^2(\mathbb{R}^2)$ with domain $H^4(\mathbb{R}^2)$. It is known that the spectrum of Δ^2 is $\sigma(\Delta^2) = [0, \infty)$. Since θ has compact support, the essential spectra of $\mathcal{L}$ and Δ^2 coincide, and so $\sigma_c(\mathcal{L}) = [0, \infty)$. We assume that θ is chosen so that $\mathcal{L}$ has a simple positive eigenvalue λ_0: