Available online at www.sciencedirect.com
ScienceDirect
Journal of Functional Analysis 260 (2011) 541–565
JOURNAL OF Functional Analysis
www.elsevier.com/locate/jfa

Two-state free Brownian motions

Michael Anshelevich [1]

Department of Mathematics, Texas A&M University, College Station, TX 77843-3368, United States

Received 15 June 2010; accepted 10 September 2010

Available online 28 September 2010

Communicated by D. Voiculescu

Abstract

In a two-state free probability space $(\mathcal{A}, \varphi, \psi)$, we define an *algebraic two-state free Brownian motion* to be a process with two-state freely independent increments whose two-state free cumulant generating function $R^{\varphi,\psi}(z)$ is quadratic. Note that a priori, the distribution of the process with respect to the second state ψ is arbitrary. We show, however, that if $\mathcal{A}$ is a von Neumann algebra, the states φ, ψ are normal, and φ is faithful, then there is only a one-parameter family of such processes. Moreover, with the exception of the actual free Brownian motion (corresponding to $\varphi = \psi$), these processes only exist for finite time.

Keywords: Free probability; Free Brownian motion; Two-state non-commutative probability space; Free stochastic integral

1. Introduction

The study of free probability was initiated by Voiculescu in the early 1980s [24]. While free probability has crucial applications to the study of operator algebras and random matrices, it has also developed into a deep and sophisticated theory in its own right. As one illustration, consider the free Central Limit Theorem. Its formulation is the same as for the usual CLT, with two changes. First, the objects involved are non-commutative random variables, that is, elements of a non-commutative $*$-algebra (or C^*-algebra, or von Neumann algebra $\mathcal{A}$, or the algebra of operators affiliated to it), with a state φ which replaces the expectation functional. Second,

E-mail address: manshel@math.tamu.edu.
[1] This work was supported in part by NSF grant DMS-0900935.

0022-1236/$ – see front matter
doi:10.1016/j.jfa.2010.09.004

independence is replaced by Voiculescu's free independence, which is more appropriate for non-commuting objects. The algebraic version of the theorem was proved in [24], followed by the full analytic version for identically distributed triangular arrays in [21] and general triangular arrays in [18]. Note that in the analytic theorems, the hypothesis on the distributions are identical to those in the usual CLT. On the other hand, in [9] and [26] the authors showed that the mode of convergence in the free CLT is actually much stronger than the classical convergence in distribution. In all these results, the limiting distribution is the semicircle law. It is characterized by having zero free cumulants of order greater than 2 (of course, in most of these results, no a priori assumption on the existence of free cumulants is made).

An important point about free probability is that, as mentioned above, there are different settings in which the theory can be studied. Consider the notion of (reduced) free product, related to the discussion above by the property that different components in a free product are freely independent. One can take a reduced free product of $*$-algebras with states, or C^*-algebras with representations, or of Hilbert spaces, or of von Neumann algebras with states, and all of these constructions are consistent. One can frequently extend purely algebraic results to the more analytic context of normed algebras (although, as illustrated in [8], such extensions are often non-trivial).

This paper is about a related theory where that is no longer the case. In [12,13], Bożejko, Leinert, and Speicher constructed what they called a conditionally free probability theory, which we will refer to as two-state free probability theory. The setting is now a $*$-algebra (von Neumann algebra, etc.) $\mathcal{A}$ with *two* states, say φ and ψ. Initially the authors had a single example of such a structure, but the theory has since been quite successful, at least in two settings. For results concerning single distributions, including the study of limit theorems, see [19,6,25]; on the other hand, for results in the purely algebraic setting, see [22,11,2]. However, very little work on this theory has been done in the analytic setting; in fact, we are only aware of one article [23]. We show here that this is not a coincidence, by the following example. We define what is natural to call (algebraic) two-state free Brownian motions. This is a very large class of processes, since the "Brownian motion" property only determines the relative position of the expectations φ and ψ, but the choice of ψ is arbitrary, at least in the algebraic setting. We then show that if $\mathcal{A}$ is a von Neumann algebra, and the expectation φ is faithful, then out of this infinite-dimensional family only a one-parameter family of processes can actually be realized. Moreover, with the exception of the actual free Brownian motion (corresponding to the case $\varphi = \psi$), these processes only exist on a finite time interval.

The paper is organized as follows. After the introduction and a background section, in Section 3 we define the two-state free Brownian motions, and show that if φ is faithful, only a one-parameter family of these processes may exist. The method of proof involves stochastic integration. In Section 4 we show that this one-parameter family actually does exist, by using a Fock space construction. We show that these processes are not Markov, even though they have classical versions, the time-reversed free Poisson processes of [16]. We also compute the generators of these processes. Finally, Section 5 contains some comments on the case when $\mathcal{A}$ is a C^*-rather than a von Neumann algebra. In particular, in this section we give another characterization of the one-parameter family mentioned above: in a large class, these are the only processes whose higher variation processes converge to the appropriate limits in $L^\infty(\varphi)$ rather than just in $L^2(\varphi)$.