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The spatial product of Arveson systems is intrinsic

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Abstract

We prove that the spatial product of two spatial Arveson systems is independent of the choice of the reference units. This also answers the same question for the minimal dilation of the Powers sum of two spatial CP-semigroups: It is independent up to cocycle conjugacy.

Keywords: Continuous product systems of Hilbert spaces; Spatial E_0-semigroups

1. Introduction

Arveson [1] associated with every ***E_0-semigroup*** (a semigroup of unital endomorphisms) on $\mathscr{B}(H)$ its ***Arveson system*** (a family of Hilbert spaces $\mathscr{E} = (\mathscr{E}_t)_{t\geqslant 0}$ with an associative identification $\mathscr{E}_s \otimes \mathscr{E}_t = \mathscr{E}_{s+t}$). He showed that E_0-semigroups are classified by their Arveson system up to cocycle conjugacy. By a ***spatial*** Arveson system we understand a pair $(\mathscr{E}, u)$ of an Arveson system $\mathscr{E}$ and a ***unital unit*** u (that is a section $u = (u_t)_{t\geqslant 0}$ of unit vectors $u_t \in \mathscr{E}_t$ that factor

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as $u_s \otimes u_t = u_{s+t}$). Spatial Arveson systems have an index, and this index is additive under the tensor product of Arveson systems.

Much of this can be carried through also for product systems of Hilbert modules and E_0-semigroups on $\mathscr{B}^a(E)$, the algebra of all adjointable operators on a Hilbert module; see the conclusive paper Skeide [19] and its list of references. However, there is no such thing as the tensor product of product systems of Hilbert modules. To overcome this, Skeide [18] (preprint, 2001) introduced the product of spatial product systems (henceforth, the ***spatial product***), under which the index of spatial product systems of Hilbert modules is additive.

It is known that the spatial structure of a spatial Arveson system $(\mathscr{E}_t)_{t\geqslant 0}$ depends on the choice of the ***reference unit*** $(u_t)_{t\geqslant 0}$. In fact, Tsirelson [22] showed that if $(v_t)_{t\geqslant 0}$ is another unital unit, then there need not exist an automorphism of $(\mathscr{E}_t)_{t\geqslant 0}$ that sends $(u_t)_{t\geqslant 0}$ to $(v_t)_{t\geqslant 0}$. Also the spatial product depends *a priori* on the choice of the reference units of its factors. This immediately raises the question if different choices of references units give isomorphic products or not. In these notes we answer this question in the affirmative sense for the spatial product of Arveson systems.

For two Arveson systems $(\mathscr{E}_t)_{t\geqslant 0}$ and $(\mathscr{F}_t)_{t\geqslant 0}$ with reference units $(u_t)_{t\geqslant 0}$ and $(v_t)_{t\geqslant 0}$, respectively, their spatial product can be identified with the subsystem of the tensor product generated by the subsets $u_t \otimes \mathscr{F}_t$ and $\mathscr{E}_t \otimes v_t$. This raises another question, namely, if that subsystem is all of the tensor product or not. This has been answered in the negative sense by Powers [13], resolving the same question for a related problem. Let us describe this problem very briefly.

Suppose we have two E_0-semigroups $\vartheta^i = (\vartheta^i_t)_{t\geqslant 0}$ on $\mathscr{B}(H^i)$ with ***intertwining*** semigroups $(U^i_t)_{t\geqslant 0}$ of isometries in $\mathscr{B}(H^i)$ (that is, $\vartheta^i_t(a^i)U_t = U_t a^i$). Intertwining semigroups correspond one-to-one with unital units of the associated Arveson systems $(\mathscr{E}^i_t)_{t\geqslant 0}$, so that these are spatial. Then by

$$T\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} := \begin{pmatrix} \vartheta^1_t(a_{11}) & U^{1*}_t a_{12} U^2_t \\ U^{2*}_t a_{21} U^1_t & \vartheta^2_t(a_{22}) \end{pmatrix}$$

we define a Markov semigroup on $\mathscr{B}(H^1 \oplus H^2)$. Its unique ***minimal dilation*** (see Bhat [5]) is an E_0-semigroup (fulfilling some properties). At the 2002 Workshop Advances in Quantum Dynamics in Mount Holyoke, Powers asked for the cocycle conjugacy class (that is, for the Arveson system) of that E_0-semigroup. More precisely, he asked if it is the cocycle conjugacy class of the tensor product of ϑ^1 and ϑ^2, or not. Still during the workshop Skeide (see the proceedings [17]) identified the Arveson system of that ***Powers sum*** as the spatial product of the Arveson systems of ϑ^1 and ϑ^2. So, Powers' question is equivalent to the question if the spatial product is the tensor product, or not.

In [13] Powers answered the former question in the negative sense and, henceforth, also the latter. He left open the question if the cocycle conjugacy class of the minimal dilation of the Powers sum depends on the choice of the intertwining isometries. Our result of the present notes tells, no, it doesn't depend. We should say that Powers in [13] to some extent considered the Powers sum not only for E_0-semigroups but also for those CP-semigroups he called as *spatial*. We think that his definition of spatial CP-semigroup is too restrictive, and prefer to use Arveson's definition [2], which is much wider; see Bhat, Liebscher, and Skeide [6]. The definition of Powers sum easily extends to those CP-semigroups and the relation of the associated Arveson system of the minimal dilations is stills the same: The Arveson system of the sum is the spatial product of the Arveson systems of the addends; see Skeide [20]. Therefore, our result here also applies to the more general situation.