Available online at www.sciencedirect.com
ScienceDirect
Journal of Functional Analysis 260 (2011) 2767–2814
JOURNAL OF Functional Analysis
www.elsevier.com/locate/jfa

Spectral theory for commutative algebras of differential operators on Lie groups

Alessio Martini

Scuola Normale Superiore, Piazza dei Cavalieri 7, 56126 Pisa, Italy

Received 29 July 2010; accepted 14 January 2011
Available online 1 February 2011
Communicated by P. Delorme

Abstract

The joint spectral theory of a system of pairwise commuting self-adjoint left-invariant differential operators $L_1, \dots, L_n$ on a connected Lie group G is studied, under the hypothesis that the algebra generated by them contains a "weighted subcoercive operator" of ter Elst and Robinson (1998) [52]. The joint spectrum of $L_1, \dots, L_n$ in every unitary representation of G is characterized as the set of the eigenvalues corresponding to a particular class of (generalized) joint eigenfunctions of positive type of $L_1, \dots, L_n$. Connections with the theory of Gelfand pairs are established in the case $L_1, \dots, L_n$ generate the algebra of K-invariant left-invariant differential operators on G for some compact subgroup K of $\mathrm{Aut}(G)$.

Keywords: Functional calculus; Differential operators; Lie groups; Joint spectrum; Eigenfunction expansions; Representation theory; Gelfand pairs

1. Introduction

Let $L_1, \dots, L_n$ be pairwise commuting smooth linear differential operators on a smooth manifold X, which are formally self-adjoint with respect to some smooth measure μ. Do these operators admit a joint functional calculus on $L^2(X, \mu)$? In that case, what is the relationship between the joint L^2 spectrum of $L_1, \dots, L_n$ and their joint smooth (possibly non-L^2) eigenfunctions on X?

A joint functional calculus for $L_1, \dots, L_n$ is given, via spectral integration, by a *joint spectral resolution* E, i.e., a resolution of the identity of $L^2(X, \mu)$ on $\mathbb{R}^n$ such that

E-mail address: a.martini@sns.it.

0022-1236/$ – see front matter
doi:10.1016/j.jfa.2011.01.008

$$\int_{\mathbb{R}^n} \lambda_j \, dE(\lambda_1, \dots, \lambda_n)$$

is a self-adjoint extension of L_j for $j = 1, \dots, n$. Existence and uniqueness of E are related to the so-called "domain problems", such as essential self-adjointness of $L_1, \dots, L_n$ and strong commutativity of their self-adjoint extensions.

Once a joint spectral resolution E is fixed, the theory of eigenfunction expansions (see, e.g., [5,39]) yields the existence, for E-almost every $\lambda = (\lambda_1, \dots, \lambda_n)$ in the joint L^2 spectrum $\Sigma = \operatorname{supp} E$ of $L_1, \dots, L_n$, of a corresponding generalized joint eigenfunction ϕ, which (under some hypoellipticity hypothesis on $L_1, \dots, L_n$) belongs to the space $\mathcal{E}(X)$ of smooth functions on X and satisfies

$$L_j \phi = \lambda_j \phi \quad \text{for } j = 1, \dots, n. \tag{1.1}$$

However, from the general theory, neither it is clear for which $\lambda \in \Sigma$ there does exist a corresponding smooth eigenfunction ϕ, nor for which $\phi \in \mathcal{E}(X)$ satisfying (1.1) the corresponding λ does belong to Σ.

In this paper, we restrict to the case of $X = G$ being a connected Lie group, with right Haar measure μ, and left-invariant differential operators $L_1, \dots, L_n$. In this context, the problem of existence and uniqueness of a joint spectral resolution can be stated for the operators $d\varpi(L_1), \dots, d\varpi(L_n)$ in every unitary representation ϖ of G — the case of the operators $L_1, \dots, L_n$ on $L^2(G)$ corresponding to the (right) regular representation of G — with a possibly different joint spectrum Σ_ϖ for each representation ϖ.

Via techniques due to Nelson and Stinespring [45], we show in Section 3.1 that a sufficient condition for the essential self-adjointness and the existence of a joint spectral resolution in every unitary representation is that the algebra generated by $L_1, \dots, L_n$ contains a *weighted subcoercive operator*. This class of hypoelliptic left-invariant differential operators, defined by ter Elst and Robinson [52] in terms of a *homogeneous contraction* of the Lie algebra $\mathfrak{g}$ of G, is large enough to contain positive elliptic operators, sublaplacians and positive Rockland operators (see Section 2 for details).

Under the same hypotheses on $L_1, \dots, L_n$, we prove that *every* element of the joint spectrum Σ corresponds to a joint (smooth) eigenfunction ϕ of $L_1, \dots, L_n$ which is a function of *positive type* on G, i.e., of the form

$$\phi(x) = \langle \pi(x)v, v \rangle \tag{1.2}$$

for some unitary representation π of G on a Hilbert space $\mathcal{H}$ and some cyclic vector $v \in \mathcal{H} \setminus \{0\}$. More precisely, in Section 4 we show that:

(a) for every unitary representation ϖ of G, Σ_ϖ coincides with the set of the eigenvalues relative to the joint eigenfunctions of $L_1, \dots, L_n$ of the form (1.2) with π (irreducible and) *weakly contained* in ϖ;
(b) if G is amenable, then Σ coincides with the set of the eigenvalues relative to *all* the joint eigenfunctions of positive type;
(c) if $L^1(G)$ is a symmetric Banach $*$-algebra, then Σ coincides with the set of the eigenvalues relative to all the *bounded* joint eigenfunctions.