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Standing waves of nonlinear Schrödinger equations with the gauge field

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Abstract

We study standing waves for nonlinear Schrödinger equations with the gauge field. Some existence results of standing waves are established by applying variational methods to the functional which is obtained by representing the gauge field A_μ in terms of complex scalar field ϕ. We also show that there exists no standing wave for certain range of parameters by establishing a new inequality of Sobolev type.

Keywords: Schrödinger; Nonlocal potential; Standing waves; Variational methods

1. Introduction and statement of main results

In this study we are interested in the existence of standing waves for the following nonlinear Schrödinger system

$$iD_0\phi + (D_1D_1 + D_2D_2)\phi = -\lambda|\phi|^{p-2}\phi, \tag{1.1}$$

$$\partial_0 A_1 - \partial_1 A_0 = -\operatorname{Im}(\bar{\phi} D_2\phi), \tag{1.2}$$

$$\partial_0 A_2 - \partial_2 A_0 = \operatorname{Im}(\bar{\phi} D_1\phi), \tag{1.3}$$

$$\partial_1 A_2 - \partial_2 A_1 = -1/2|\phi|^2, \tag{1.4}$$

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where i denotes the imaginary unit, $\partial_0 = \frac{\partial}{\partial t}$, $\partial_1 = \frac{\partial}{\partial x_1}$, $\partial_2 = \frac{\partial}{\partial x_2}$ for $(t, x_1, x_2) \in \mathbb{R}^{1+2}$, $\phi : \mathbb{R}^{1+2} \to \mathbb{C}$ is the complex scalar field, $A_\mu : \mathbb{R}^{1+2} \to \mathbb{R}$ is the gauge field, $D_\mu = \partial_\mu + iA_\mu$ is the covariant derivative for $\mu = 0, 1, 2$, and $\lambda > 0$ is a constant representing the strength of interaction potential. We consider only the superlinear case $p > 2$. The system (1.1)–(1.4) proposed in [10,11] consists of the Schrödinger equation augmented by the gauge field A_μ.

A special case with $p = 4$, $\lambda = 1$ has received much attention and has been studied by several authors, where one can derive the following self-dual equations (see [9,11])

$$\begin{aligned} &D_1\phi + iD_2\phi = 0, \qquad A_0 = \frac{1}{2}|\phi|^2, \\ &\partial_1 A_2 - \partial_2 A_1 = -1/2\,|\phi|^2, \qquad \partial_1 A_1 + \partial_2 A_2 = 0. \end{aligned} \tag{1.5}$$

Then the self-dual equations (1.5) can be transformed into the Liouville equation, an integrable equation whose solutions are explicitly known. We note that solutions to the self-dual equations (1.5) provide static solutions to Eqs. (1.1)–(1.4) with $p = 4$ and $\lambda = 1$. For more information on the self-dual equations, we refer to [7].

In this paper, we seek the standing wave solutions to (1.1)–(1.4) for $p > 2$ of the form

$$\begin{aligned} &\phi(t, x) = u\big(|x|\big)e^{i\omega t}, \qquad A_0(t, x) = k\big(|x|\big), \\ &A_1(t, x) = \frac{x_2}{|x|^2}h\big(|x|\big), \quad A_2(t, x) = -\frac{x_1}{|x|^2}h\big(|x|\big), \end{aligned} \tag{1.6}$$

where $\omega > 0$ is a given frequency and u, k, h are real valued functions on $[0, \infty)$ such that $h(0) = 0$. We are looking for the classical solution, that is, a solution (ϕ, A_0, A_1, A_2) of (1.1)–(1.4) in the class $C^2(\mathbb{R}^2) \times C^1(\mathbb{R}^2) \times C^1(\mathbb{R}^2) \times C^1(\mathbb{R}^2)$. We point out that the ansatz (1.6) satisfies the Coulomb gauge condition $\partial_1 A_1 + \partial_2 A_2 = 0$. Inserting the ansatz (1.6) into the system of Eqs. (1.1)–(1.4), we get the following nonlocal semi-linear elliptic equation for u

$$\Delta u - \omega u - \left(\xi + \int_{|x|}^{\infty} \frac{h(s)}{s}u^2(s)\,ds\right)u - \frac{h^2(|x|)}{|x|^2}u + \lambda|u|^{p-2}u = 0 \quad \text{in } \mathbb{R}^2, \tag{1.7}$$

where $h(s) = \int_0^s \frac{l}{2}u^2(l)\,dl$ and $\xi \in \mathbb{R}$ is a constant (see Section 2 for a choice of ξ). We will show that (1.7) is actually the Euler–Lagrange equation of the functional

$$J(u) \equiv \frac{1}{2}\int_{\mathbb{R}^2} |\nabla u|^2 + (\omega + \xi)u^2 + \frac{u^2}{|x|^2}\left(\int_0^{|x|} \frac{s}{2}u^2(s)\,ds\right)^2 dx - \frac{\lambda}{p}\int_{\mathbb{R}^2} |u|^p\,dx, \quad u \in H_r^1.$$

Here H_r^1 denotes the set of radially symmetric functions in $H^1(\mathbb{R}^2)$. We will show that $J \in C^1(H_r^1)$ and a critical point u of J produces a standing wave (ϕ, A_0, A_1, A_2) of the form (1.6).

If $p > 4$, the functional J has the mountain pass structure when $\omega + \xi > 0$. When we apply directly the mountain pass theorem [2] to get a critical point of J, it is important to check whether Palais–Smale condition holds, that is, whether there exists a convergent subsequence for any sequence $\{u_n\}_n$ satisfying $\lim_{n\to\infty} J'(u_n) = 0$ and $\lim_{n\to\infty} J(u_n) \in \mathbb{R}$. For $p \geqslant 6$, it is standard to show that Palais–Smale condition holds for J. On the other hand, for $p \in (4, 6)$, we do not