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Reducing subspaces for analytic multipliers of the Bergman space ☆

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Abstract

In Douglas et al. (2011) [4] some incisive results are obtained on the structure of the reducing subspaces for the multiplication operator M_ϕ by a finite Blaschke product ϕ on the Bergman space on the unit disk. In particular, the linear dimension of the commutant, $\mathcal{A}_\phi = \{M_\phi, M_\phi^*\}'$, is shown to equal the number of connected components of the Riemann surface, $\phi^{-1} \circ \phi$. Using techniques from Douglas et al. (2011) [4] and a uniformization result that expresses ϕ as a holomorphic covering map in a neighborhood of the boundary of the disk, we prove that $\mathcal{A}_\phi$ is commutative, and moreover, that the minimal reducing subspaces are pairwise orthogonal. Finally, an analytic/arithmetic description of the minimal reducing subspaces is also provided, along with the taxonomy of the possible structures of the reducing subspaces in case ϕ has eight zeros. These results have implications in both operator theory and the geometry of finite Blaschke products.

Keywords: Reducing subspace; Bergman space; Finite Blaschke product

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1. Introduction

The present article is a continuation of [4] and a series of recent related works, such as [5,6,8]. Our aim is to classify the reducing subspaces of analytic Toeplitz operators with a rational, inner symbol acting on the Bergman space of the unit disk. While a similar study in the case of the Hardy space was completed a long time ago (see [3,12,13]), investigation of the Bergman space setting was started only a few years ago. Not surprisingly, the structure and relative position of these reducing subspaces in the Bergman space reveal a rich geometric (Riemann surface) picture directly dependent on the rational symbol of the Toeplitz operator.

We start by recalling a few basic facts and some terminology. The Bergman space $L_a^2(\mathbb{D})$ is the space of holomorphic functions on $\mathbb{D}$ which are square-integrable with respect to the Lebesgue measure dm on $\mathbb{D}$. For a bounded holomorphic function ϕ on the unit disk, the multiplication operator, $M_\phi : L_a^2(\mathbb{D}) \to L_a^2(\mathbb{D})$, is defined by

$$M_\phi(h) = \phi h, \quad h \in L_a^2(\mathbb{D}).$$

The Toeplitz operator T_ϕ on $L_a^2(\mathbb{D})$ with symbol $\phi \in L^\infty(\mathbb{D})$ acts as

$$T_\phi(h) = P(\phi h), \quad h \in L_a^2,$$

where P is the orthogonal projection from $L^2(\mathbb{D})$ to $L_a^2(\mathbb{D})$. Note that $T_\phi = M_\phi$ whenever ϕ is holomorphic.

An invariant subspace $\mathcal{M}$ for M_ϕ is a closed subspace of $L_a^2(\mathbb{D})$ satisfying $\phi\mathcal{M} \subseteq \mathcal{M}$. If, in addition, $M_\phi^*\mathcal{M} \subseteq \mathcal{M}$, we call $\mathcal{M}$ a reducing subspace of M_ϕ. We say $\mathcal{M}$ is a minimal reducing subspace if there is no nontrivial reducing subspace for M_ϕ contained in $\mathcal{M}$. The study of invariant subspaces and reducing subspaces for various classes of linear operators has inspired much deep research and prompted many interesting problems. Even for the multiplication operator M_z, the lattice of invariant subspaces of $L_a^2(\mathbb{D})$ is huge and its order structure remains a mystery. Progress in understanding the lattice of reducing subspaces of M_ϕ was only recently made, and only in the case of inner function symbols [4–8,10,11,14].

Let $\{M_\phi\}' = \{X \in \mathscr{L}(L_a^2(\mathbb{D})):\ M_\phi X = X M_\phi\}$ be the commutant algebra of M_ϕ. The problem of classifying the reducing subspaces of M_ϕ is equivalent to finding the projections in $\{M_\phi\}'$. This classification problem in the case of the Hardy space was the motivation of the highly original works by Thomson and Cowen (see [3,12,13]). They used the Riemann surface of $\phi^{-1} \circ \phi$ as a basis for the description of the commutant of M_ϕ acting on the Hardy space. A notable fact for our study is that inner function symbols played a dominant role in their studies. In complete analogy, in the Bergman space $L_a^2(\mathbb{D})$ framework, one can essentially use the same proof to show that for a "nice" analytic function f, there exists a finite Blaschke product ϕ such that $\{M_f\}' = \{M_\phi\}'$. Therefore, the structure of the reducing subspaces of the multiplier M_f on the Bergman space of the disk is the same as that for M_ϕ.

Zhu showed in [14] that for each Blaschke product of order 2, there exist exactly 2 different minimal reducing subspaces of M_ϕ. This result also appeared in [10]. Zhu also conjectured in [14] that M_ϕ has exactly n distinct minimal reducing subspaces for a Blaschke product ϕ of order n. The results in [8] disproved Zhu's conjecture, and the authors raised a modification in which M_ϕ was conjectured to have at most n distinct minimal reducing subspaces for a Blaschke product ϕ of order n. Some partial results on this conjecture were obtained in [5,8,11]. These authors proved the finiteness result in case $n \leqslant 6$, each using a different method. A notable result