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On the Cauchy problem for the periodic generalized Degasperis–Procesi equation

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Abstract

We mainly study the Cauchy problem of the periodic generalized Degasperis–Procesi equation. First, we establish the local well-posedness for the equation. Second, we give the precise blow-up scenario, a conservation law and prove that the equation has smooth solutions which blow up in finite time. Finally, we investigate the blow-up rate for the blow-up solutions.

Keywords: The periodic generalized Degasperis–Procesi equation; Well-posedness; Blow-up scenario; Blow-up phenomena; Conservation law; Blow-up rate

1. Introduction

In this paper, we consider the Cauchy problem for the periodic generalized Degasperis–Procesi equation:

$$\begin{cases} u_t - u_{txx} + 4f'(u)u_x = 3f''(u)u_x u_{xx} + f'''(u)u_x^3 + f'(u)u_{xxx}, \\ \qquad\qquad\qquad\qquad t > 0,\ x \in \mathbb{R}, \\ u(0,x) = u_0(x), \qquad x \in \mathbb{R}, \\ u(t,x) = u(t,x+1), \quad t > 0,\ x \in \mathbb{R}, \end{cases} \tag{1.1}$$

where $f : \mathbb{R} \to \mathbb{R}$ is a given C^m-function, $m > 3$.

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For $f(u)=\frac{u^2}{2}$ Eq. (1.1) becomes the periodic Degasperis–Procesi equation [20]:

$$\begin{cases} u_t - u_{txx} + 4uu_x = 3u_x u_{xx} + uu_{xxx}, & t>0,\ x\in\mathbb{R},\\ u(0,x)=u_0(x), & x\in\mathbb{R},\\ u(t,x)=u(t,x+1), & t>0,\ x\in\mathbb{R}. \end{cases} \tag{1.2}$$

Degasperis, Holm and Hone [20,21] proved the formal integrability of Eq. (1.2) by constructing a Lax pair. They also showed that it has bi-Hamiltonian structure and an infinite sequence of conserved quantities, and admits exact peakon solutions which are analogous to the Camassa–Holm peakons [2–5,9,18,19].

The Degasperis–Procesi equation can be regarded as a model for nonlinear shallow water dynamics and its asymptotic accuracy is the same as that for the Camassa–Holm shallow water equation [1,6,11,14–16,27,32]. Dullin, Gottwald and Holm [22] showed that the Degasperis–Procesi equation can be obtained from the shallow water elevation equation by an appropriate Kodama transformation. Vakhnenko and Parkes [34] investigated the traveling wave solutions of Eq. (1.2) and Holm and Staley [26] studied the stability of solitons and peakons [35] numerically. Lundmark and Szmigielski [30] also presented an inverse scattering approach for computing n-peakon solutions to Eq. (1.2).

After the Degasperis–Procesi equation was derived, many results were obtained [23,24,29]. Such as, Yin proved local well-posedness of Eq. (1.2) with initial data $u_0\in H^s(\mathbb{R})$, $s>\frac{3}{2}$ on the line [37] and on the circle [38]. In these two papers the precise blow-up scenario and a blow-up result were derived. The global existence of strong solutions and global weak solutions to Eq. (1.2) are also investigated in [39,40]. Recently, Lenells [28] classified all weak traveling wave solutions. Matsuno [31] studied multisoliton solutions and their peakon limits.

Analogous to the case of the Camassa–Holm equation [10], Henry [25] and Mustafa [33] showed that smooth solutions to Eq. (1.2) have infinite speed of propagation. Coclite and Karlsen [7] also obtained global existence results for entropy weak solutions belonging to the class $L^1(\mathbb{R})\cap BV(\mathbb{R})$ and the class $L^2(\mathbb{R})\cap L^4(\mathbb{R})$. The Cauchy problem for the generalized Degasperis–Procesi equation has been studied in [36].

However, the periodic generalized Degasperis–Procesi equation has not been discussed yet. The aim of this paper is to establish the local well-posedness of Eq. (1.1), to give the precise blow-up scenario, an important conservation law and to show that Eq. (1.1) has blow-up solutions, provided their initial data satisfy certain conditions. While the local well-posedness results in Section 2 are similar to the corresponding results on the line [36], some blow-up results in Section 3 use the periodicity property in an essential way [8,12,17,24].

The paper is organized as follows. In Section 2, we establish local well-posedness of Eq. (1.1). In Section 3, we derive a precise blow-up scenario and present several blow-up results of strong solutions to Eq. (1.1). In Section 4, we investigate the blow-up rate for the blow-up solutions of Eq. (1.1).

2. Local well-posedness

In the section, we will apply Kato's theory to establish the local well-posedness for the Cauchy problem of Eq. (1.1) in $H^s(\mathbb{S})$, $s>\frac{3}{2}$ with $\mathbb{S}=\mathbb{R}/\mathbb{Z}$ (the circle of unite length).

First, we introduce some notations. If A is an unbounded operator, $D(A)$ denotes the domain of the operator A, $[A,B]=AB-BA$ denotes the commutator of the linear operators A and B, $\|\cdot\|_X$ denotes the norm of the Banach space X. For convenience, let $\|\cdot\|_s$ and $(\cdot,\cdot)_s$ denote the