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Schatten–von Neumann properties in the Weyl calculus

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Abstract

Let $\mathsf{Op}_t(a)$, for $t \in \mathbf{R}$, be the pseudo-differential operator

$$f(x) \mapsto (2\pi)^{-n} \iint a\big((1-t)x + ty, \xi\big) f(y) e^{i\langle x-y,\xi\rangle}\, dy\, d\xi$$

and let $\mathscr{I}_p$ be the set of Schatten–von Neumann operators of order $p \in [1, \infty]$ on L^2. We are especially concerned with the Weyl case (i.e. when $t = 1/2$). We prove that if m and g are appropriate metrics and weight functions respectively, h_g is the Planck's function, $h_g^{k/2} m \in L^p$ for some $k \geqslant 0$ and $a \in S(m, g)$, then $\mathsf{Op}_t(a) \in \mathscr{I}_p$, iff $a \in L^p$. Consequently, if $0 \leqslant \delta < \rho \leqslant 1$ and $a \in S^r_{\rho,\delta}$, then $\mathsf{Op}_t(a)$ is bounded on L^2, iff $a \in L^\infty$.
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0. Introduction

The aim of the paper is to continue the discussions in [10,12,26] on general continuity and compactness properties for pseudo-differential operators, especially for Weyl operators, with smooth symbols which belong to certain Hörmander classes. We are especially focused on find-

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ing necessary and sufficient conditions on particular symbols in order for the corresponding pseudo-differential operators should be Schatten–von Neumann operators of certain degrees.

If V is a real vector space of finite dimension n, V' its dual space, $t \in \mathbf{R}$ is fixed and $a \in \mathscr{S}'(V \times V')$ (we use the same notation for the usual functions and distribution spaces as in [18]), then the pseudo-differential operator $\mathsf{Op}_t(a)$ of a is the continuous linear map from $\mathscr{S}(V)$ to $\mathscr{S}'(V)$ defined by

$$\mathsf{Op}_t(a)f(x) = (2\pi)^{-n} \iint\limits_{V \times V'} a\big((1-t)x + ty, \xi\big) f(y) e^{i\langle x-y,\xi\rangle}\, dy\, d\xi. \tag{0.1}$$

(In the case when a is not an integrable function, $\mathsf{Op}_t(a)$ is interpreted as the operator with Schwartz kernel equal to $(2\pi)^{-n/2}\mathscr{F}_2^{-1}a((1-t)x+ty, x-y)$, where $\mathscr{F}_2 U(x,\xi)$ denotes the partial Fourier transform $\mathscr{F}$ on $U(x,y)$ with respect to the second variable. Here $\mathscr{F}$ is the Fourier transform which takes the form

$$\mathscr{F} f(\xi) = \widehat{f}(\xi) = (2\pi)^{-n/2} \int f(x) e^{-i\langle x,\xi\rangle}\, dx, \tag{0.2}$$

when $f \in \mathscr{S}(V)$. See also Section 18.5 in [18].) The operator $\mathsf{Op}_{1/2}(a)$ is the Weyl operator of a, and is denoted by $\mathsf{Op}^w(a)$. (See $(0.1)'$ in Section 1.)

A family of symbol classes, which appears in several situations, concerns $S^r_{\rho,\delta}(\mathbf{R}^{2n})$, for $r, \rho, \delta \in \mathbf{R}$, which consists of all smooth functions a on $\mathbf{R}^{2n}$ such that

$$\big|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)\big| \leqslant C_{\alpha,\beta} \langle \xi \rangle^{r+|\alpha|\delta-|\beta|\rho}.$$

Here $\langle \xi \rangle = (1+|\xi|^2)^{1/2}$. By letting $s_{t,\infty}$ be the set of all $a \in \mathscr{S}'$ such that the definition of $\mathsf{Op}_t(a)$ extends to a continuous operator on L^2, the following is a consequence of Theorem 18.1.11 and the comments on page 94 in [18]: *Assume that* $0 \leqslant \delta \leqslant \rho \leqslant 1$ *and* $\delta < 1$. *Then* $S^r_{\rho,\delta} \subseteq s_{t,\infty}$ *if and only if* $r \leqslant 0$. The latter equivalence can also be formulated as

$$S^r_{\rho,\delta} \subseteq s_{t,\infty} \quad \Longleftrightarrow \quad S^r_{\rho,\delta} \subseteq L^\infty. \tag{0.3}$$

A similar property holds for any "reasonable" family of symbol classes. This is a consequence of the investigations in [2,3,16,18]. For example, in [16,18], Hörmander introduces a family of symbol classes, denoted by $S(m,g)$, which is parameterized by the weight function m and the Riemannian metric g. (See Section 1 for strict definition.) By choosing m and g in appropriate ways, it follows that most of those reasonable symbol classes can be obtained, e.g. $S^r_{\rho,\delta}$ is obtained in such way. If m and g are appropriate, then (0.3) is generalized into:

$$S(m,g) \subseteq s_{t,\infty} \quad \Longleftrightarrow \quad S(m,g) \subseteq L^\infty. \tag{0.3$'$}$$

(Here we remark that important contributions for improving the calculus on $S(m,g)$ can be found in [6–9]. For example in [7], Bony extends parts of the theory to a family of symbol classes which contains any $S(m,g)$ when m and g are appropriate.)

In [10,26], the equivalence $(0.3)'$ is extended in such way that it involves Schatten–von Neumann properties. More precisely, let $s_{t,p}(V \times V')$ be the set of all $a \in \mathscr{S}'(V \times V')$ such that $\mathsf{Op}_t(a)$ belongs to $\mathscr{I}_p$, the set of Schatten–von Neumann operators of order $p \in [1,\infty]$ on

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