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Sharp logarithmic inequalities for Riesz transforms

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Abstract

Let d be a given positive integer and let $\{R_j\}_{j=1}^d$ denote the collection of Riesz transforms on $\mathbb{R}^d$. For any $K > 2/\pi$ we determine the optimal constant L such that the following holds. For any locally integrable Borel function f on $\mathbb{R}^d$, any Borel subset A of $\mathbb{R}^d$ and any $j = 1, 2, \dots, d$ we have

$$\int_A \left|R_j f(x)\right| \mathrm{d}x \leqslant K \int_{\mathbb{R}^d} \Psi\left(\left|f(x)\right|\right) \mathrm{d}x + |A| \cdot L.$$

Here $\Psi(t) = (t+1)\log(t+1) - t$ for $t \geqslant 0$. The proof is based on probabilistic techniques and the existence of certain special harmonic functions. As a by-product, we obtain related sharp estimates for the so-called re-expansion operator, an important object in some problems of mathematical physics.

Keywords: Hilbert transform; Riesz transform; $L \log L$ inequality; Martingale; Differential subordination; Re-expansion operator; Best constant

1. Introduction

One of the most basic examples of Calderón–Zygmund singular integrals in $\mathbb{R}^d$ is the collection of Riesz transforms [20]:

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$$R_j f(x) = \frac{\Gamma(\frac{d+1}{2})}{\pi^{(d+1)/2}} \int_{\mathbb{R}^d} \frac{x_j - y_j}{|x-y|^{d+1}} f(y)\,\mathrm{d}y, \quad j = 1, 2, \ldots, d,$$

where the integrals are supposed to exist in the sense of Cauchy principal values. In the particular case $d = 1$, the family consists of only one element, the Hilbert transform $\mathcal{H}$ on $\mathbb{R}$. Alternatively, R_j can be defined as the Fourier multiplier with the symbol $-i\xi_j/|\xi|$, $\xi \in \mathbb{R}^d \setminus \{0\}$; that is, we have the following relation between the Fourier transforms of f and $R_j f$:

$$\widehat{R_j f}(\xi) = -i \frac{\xi_j}{|\xi|} \widehat{f}(\xi), \quad \text{for } \xi \in \mathbb{R}^d \setminus \{0\}. \tag{1}$$

It has been long of interest to study various norms of these operators. The classical result of M. Riesz [19] states that $\mathcal{H}$ is a bounded operator on $L^p(\mathbb{R})$ if and only if $1 < p < \infty$. Gokhberg and Krupnik [7] derived the precise value of the norm $\|\mathcal{H}\|_{L^p(\mathbb{R}) \to L^p(\mathbb{R})}$ for $p = 2^k$, $k = 1, 2, \ldots,$ and Pichorides [18] determined the norms for the remaining p: we have

$$\|\mathcal{H}\|_{L^p(\mathbb{R}) \to L^p(\mathbb{R})} = C_p := \begin{cases} \tan(\frac{\pi}{2p}) & \text{if } 1 < p \leqslant 2, \\ \cot(\frac{\pi}{2p}) & \text{if } p \geqslant 2. \end{cases} \tag{2}$$

Using the so-called method of rotations, Iwaniec and Martin [14] extended this result to the d-dimensional setting: they proved that for $1 < p < \infty$ and any function $f \in L^p(\mathbb{R}^d)$,

$$\|R_j f\|_{L^p(\mathbb{R}^d)} \leqslant C_p \|f\|_{L^p(\mathbb{R}^d)}, \quad j = 1, 2, \ldots, d, \tag{3}$$

and the constant C_p cannot be decreased. In other words, they showed that the norms $\|R_j\|_{L^p(\mathbb{R}^d) \to L^p(\mathbb{R}^d)}$ and $\|\mathcal{H}\|_{L^p(\mathbb{R}) \to L^p(\mathbb{R})}$ coincide. An alternative, probabilistic proof of the estimate (3), based on a sharp estimate for orthogonal martingales, was given by Bañuelos and Wang in [1].

Our motivation comes from the question about the limit case $p = 1$. Riesz transforms are not bounded on L^1, but there are several important substitutes for (3). Kolmogorov [16] proved the weak-type (1, 1) estimate

$$\left|\left\{x \in \mathbb{R}\colon \left|\mathcal{H} f(x)\right| \geqslant 1\right\}\right| \leqslant c_1 \|f\|_{L^1(\mathbb{R})}$$

for some universal constant $c_1 < \infty$. The optimal value of c_1 was found by Davis [5] to be equal to

$$\frac{1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots}{1 - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \cdots} \simeq 1.34\ldots.$$

This result was further extended by Janakiraman [15], who established the weak-type (p, p) bound

$$\left|\left\{x \in \mathbb{R}\colon \left|\mathcal{H} f(x)\right| \geqslant 1\right\}\right| \leqslant \left(\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{|\frac{2}{\pi} \log |t||^p}{t^2 + 1}\,\mathrm{d}t\right)^{-1} \|f\|_{L^p(\mathbb{R})}^p, \quad 1 \leqslant p \leqslant 2,$$