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Modified zeta functions as kernels of integral operators

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Abstract

The modified zeta functions $\sum_{n \in K} n^{-s}$, where $K \subset \mathbb{N}$, converge absolutely for $\operatorname{Re} s > 1$. These generalise the Riemann zeta function which is known to have a meromorphic continuation to all of $\mathbb{C}$ with a single pole at $s = 1$. Our main result is a characterisation of the modified zeta functions that have pole-like behaviour at this point. This behaviour is defined by considering the modified zeta functions as kernels of certain integral operators on the spaces $L^2(I)$ for symmetric and bounded intervals $I \subset \mathbb{R}$. We also consider the special case when the set $K \subset \mathbb{N}$ is assumed to have arithmetic structure. In particular, we look at local L^p integrability properties of the modified zeta functions on the abscissa $\operatorname{Re} s = 1$ for $p \in [1, \infty]$.

Keywords: Zeta function; Integral operator; Frame theory; Tauberian theory

1. Introduction

We consider the behaviour of the modified zeta functions defined by

$$\zeta_K(s) = \sum_{n \in K} \frac{1}{n^s}, \quad K \subset \mathbb{N}, \tag{1.1}$$

near the point $s = 1$. Here $s = \sigma + \mathrm{i}t$ denotes the complex variable. The infinite series defining these functions converge absolutely in the half-plane $\sigma > 1$. We refer to these as K-zeta functions. Note that for $K = \mathbb{N}$, the formula (1.1) defines the Riemann zeta function.

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The main objective of this paper is, for general $K \subset \mathbb{N}$, to find an operator-theoretic generalisation of the classical result, due to B. Riemann, that the Riemann zeta function can be expressed as

$$\zeta(s) = \frac{1}{s-1} + \psi(s), \tag{1.2}$$

where ψ is an entire function. (See [5] for an extensive discussion, as well as an English translation, of Riemann's original paper.)

To motivate this approach, we note that although N. Kurokawa [16] found sufficient conditions on the sets K for ζ_K to have an analytic continuation across the abscissa $\sigma = 1$, it was shown by J.-P. Kahane and H. Queffelec [11,26] that for most choices of the subset K, in the sense of Baire categories, the K-zeta functions have the abscissa $\sigma = 1$ as a natural boundary. So, instead of looking at the formula (1.2) as a statement about analytic continuation, we consider it as saying that the local behaviour of $\zeta(s)$ at $s = 1$ is an analytic, and therefore small, perturbation of a pole with residue one.

To interpret this in operator-theoretic terms, we define, for $K \subset \mathbb{N}$ and intervals I of the form $(-T, T)$ with finite $T > 0$, the family of operators

$$\mathcal{Z}_{K,I} : g \in L^2(I) \longmapsto \lim_{\delta \to 0} \frac{\chi_I(t)}{\pi} \int_I g(\tau) \operatorname{Re} \zeta_K\big(1 + \delta + \mathrm{i}(t - \tau)\big)\, \mathrm{d}\tau \in L^2(I).$$

Here the characteristic function χ_I is applied to emphasise that we look at $L^2(I)$ as a subspace of $L^2(\mathbb{R})$. To understand these operators, we consider the example $K = \mathbb{N}$. The formula (1.2) implies

$$\operatorname{Re} \zeta_{\mathbb{N}}(1 + \delta + \mathrm{i}t) = \frac{\delta}{\delta^2 + t^2} + \operatorname{Re} \psi(1 + \delta + \mathrm{i}t),$$

whence

$$\mathcal{Z}_{\mathbb{N},I} = \mathrm{Id} + \Psi_{\mathbb{N},I}, \tag{1.3}$$

for a compact operator $\Psi_{\mathbb{N},I}$ and the identity operator Id. Indeed, the term $\pi^{-1}\delta/(\delta^2 + t^2)$ is the Poisson kernel which, under convolution, gives rise to the identity operator, while convolution with continuous kernels give compact operators (see Lemma 2). Hence, $\mathcal{Z}_{\mathbb{N},I}$ is a compact, and therefore a small perturbation of the identity operator.

In Theorem 1, we generalise the above formula in the following manner. We show that given $K \subset \mathbb{N}$, and a bounded and symmetric interval $I \subset \mathbb{R}$, there exist a subset $L \subset \mathbb{R}$ and a compact operator $\Phi_{K,I}$ such that

$$\mathcal{Z}_{K,I} = \chi_I \mathcal{F}^{-1} \chi_L \mathcal{F} + \Phi_{K,I}.$$

We remark that, intuitively, large $K \subset \mathbb{N}$ should correspond to large $L \subset \mathbb{R}$. In fact, it follows from our construction (see (2.1) below) that if $K = \mathbb{N}$ then $L = \mathbb{R}$. Hence, $\chi_I \mathcal{F}^{-1} \chi_L \mathcal{F} = \mathrm{Id}$ and we obtain again the formula (1.3).