Available online at www.sciencedirect.com
ScienceDirect
Journal of Functional Analysis 259 (2010) 503–523
JOURNAL OF Functional Analysis
www.elsevier.com/locate/jfa

Positive commutators at the bottom of the spectrum ☆

András Vasy [a,b], Jared Wunsch [a,b,*]

[a] Department of Mathematics, Stanford University, United States
[b] Department of Mathematics, Northwestern University, Evanston, IL, United States

Received 20 October 2009; accepted 12 April 2010

Communicated by D. Voiculescu

Abstract

Bony and Häfner have recently obtained positive commutator estimates on the Laplacian in the low-energy limit on asymptotically Euclidean spaces; these estimates can be used to prove local energy decay estimates if the metric is non-trapping. We simplify the proof of the estimates of Bony–Häfner and generalize them to the setting of scattering manifolds (i.e. manifolds with large conic ends), by applying a sharp Poincaré inequality. Our main result is the positive commutator estimate

$$\chi_I(H^2\Delta_g)\frac{i}{2}[H^2\Delta_g, A]\chi_I(H^2\Delta_g) \geqslant C\chi_I(H^2\Delta_g)^2,$$

where $H \uparrow \infty$ is a *large* parameter, I is a compact interval in $(0,\infty)$, and χ_I its indicator function, and where A is a differential operator supported outside a compact set and equal to $(1/2)(rD_r + (rD_r)^*)$ near infinity. The Laplacian can also be modified by the addition of a positive potential of sufficiently rapid decay—the same estimate then holds for the resulting Schrödinger operator.

Keywords: Low energy; Commutator; Mourre; Energy decay

☆ The authors thank Rafe Mazzeo for helpful conversations and pointing out the reference Mazzeo and McOwen (2001) [6]; an anonymous referee also provided helpful comments on the exposition. They gratefully acknowledge partial support from the NSF under grant numbers DMS-0801226 (A.V.) and DMS-0700318 (J.W.).

* Corresponding author at: Department of Mathematics, Northwestern University, United States.
E-mail addresses: andras@math.stanford.edu (A. Vasy), jwunsch@math.northwestern.edu (J. Wunsch).

0022-1236/$ – see front matter
doi:10.1016/j.jfa.2010.04.012

1. Introduction

The purpose of this paper is to clarify an intricate argument recently introduced by Bony and Häfner [1] and use these ideas to generalize certain of the results of [1]. The central thrust of [1] is first of all to obtain certain kinds of commutator estimates for the Laplacian and its square root on asymptotically Euclidean space. The authors then employ those estimates to yield energy decay results for the wave equation, and, ultimately, global existence results for quadratically semilinear wave equations on these spaces. In a subsequent note [2], applications of the linear results to the low frequency limiting absorption principle were shown. The novel tool central to all of these applications is the commutator estimate

$$\chi_I\big(H^2\Delta_g\big)\frac{i}{2}\big[H^2\Delta_g, A\big]\chi_I\big(H^2\Delta_g\big) \geqslant C\chi_I\big(H^2\Delta_g\big)^2, \tag{1.1}$$

where $H \uparrow \infty$ is a *large* parameter, I is a compact interval in $(0,\infty)$, and χ_I its indicator function, and where A is a differential operator supported outside a compact set and equal to $(1/2)(rD_r + (rD_r)^*)$ near infinity. The estimate (1.1) is thus a low-energy version of the positive commutator construction that is ubiquitous in scattering theory; we remark that the analogous *high*-energy estimate would not be true with this choice of A, supported outside a compact set: by standard results in microlocal analysis, the symbol of A would have to be strictly increasing along all geodesics, lifted to the cotangent bundle. Indeed, on a manifold with trapped geodesics, the construction of such a high-energy commutant is manifestly impossible.

In [1], the estimate (1.1) is proved by a multi-step process involving a sequence of perturbation arguments, starting from flat $\mathbb{R}^n$. It is thus a priori unclear whether such estimates continue to hold if we vary the topology of our space and its end structure. In this paper we show that (1.1) (as well as a related estimate for $\sqrt{\Delta}$) does indeed continue to hold on any long-range metric perturbation of a *scattering manifold*, and further holds even if a short range (in a suitable sense) non-negative potential is added. The class of scattering manifolds, introduced by Melrose [8], consists of all manifolds with ends that look asymptotically like the large ends of cones. The topology of interior and of the cross sections of the ends is unrestricted. Our methods are non-perturbative and simple, involving only commutator estimates for differential operators and a sharp Poincaré-type inequality on these manifolds. We anticipate that these methods will prove quite flexible in the investigation of energy decay in a variety of other asymptotic geometries.

We do not explore the applications of our estimate in detail here, as the methods of [1] apply, mutatis mutandis, directly to our situation. We content ourselves with restating the energy decay estimate of [1] for solutions to the wave equation in the final section of the paper and sketching the main ingredients in its proof, adapted to our setting. This estimate applies on scattering manifolds with no trapped geodesics.[1]

We point out here that Guillarmou and Hassell started an extensive and very detailed study of the Laplacian on scattering manifolds near the bottom of the spectrum, [4], with a particular emphasis on the Schwartz kernel of the resolvent of the Laplacian on a resolved space. Our methods give the estimates we need more quickly, but naturally the results of [4] give more detail on the resolvent kernel, which in principle implies for instance results on the energy decay.[2] We

[1] Such a manifold must in fact be contractible, but we note that even $\mathbb{R}^n$ can be equipped with scattering metrics different from the round metric on the sphere at infinity, so this result remains broader than that of [1].

[2] Note, however, that L^2-based estimates are not always easy to get from precise description of the Schwartz kernel!