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Weak spectral synthesis in commutative Banach algebras. II

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Abstract

Let A be a semisimple and regular commutative Banach algebra with structure space $\Delta(A)$. Continuing our investigation in [E. Kaniuth, Weak spectral synthesis in commutative Banach algebras, J. Funct. Anal. 254 (2008) 987–1002], we establish various results on intersections and unions of weak spectral sets and weak Ditkin sets in $\Delta(A)$. As an important example, the algebra of n-times continuously differentiable functions is studied in detail. In addition, we prove a theorem on spectral synthesis for projective tensor products of commutative Banach algebras which applies to Fourier algebras of locally compact groups.

Keywords: Commutative Banach algebra; Structure space; Weak spectral set; Weak Ditkin set; Projective tensor product; Fourier algebra; Algebra of continuously differentiable functions

0. Introduction

Let A be a regular and semisimple commutative Banach algebra with structure space $\Delta(A)$ and Gelfand transform $a \to \widehat{a}$. For any subset M of A, the hull $h(M)$ of M is defined by $h(M) = \{\varphi \in \Delta(A)\colon\ \varphi(M) = \{0\}\}$. Associated to each closed subset E of $\Delta(A)$ are two distinguished ideals with hull equal to E, namely

$$k(E) = \left\{a \in A\colon\ \widehat{a}(\varphi) = 0 \text{ for all } \varphi \in E\right\}$$

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and

$$j(E) = \{a \in A\colon \widehat{a} \text{ has compact support disjoint from } E\}.$$

Then $k(E)$ is the largest ideal with hull E and $j(E)$ is the smallest such ideal. The set E is called a *weak spectral set* (respectively, a *weak Ditkin set*) if there exists $n \in \mathbb{N}$ such that $a^n \in \overline{j(E)}$ (respectively, $a^n \in \overline{a^n j(E)}$) for every $a \in k(E)$. The smallest such number n is denoted $\xi(E)$ (respectively, $\eta(E)$). Thus $\xi(E) = 1$ ($\eta(E) = 1$) if and only if E is a set of synthesis (a Ditkin set). We say that *weak spectral synthesis* holds for A if $\xi(E) < \infty$ for every closed subset E of $\Delta(A)$.

Weak spectral sets were introduced and studied by Warner [24] in connection with the union problem for sets of synthesis. However, they appeared implicitly earlier in the work of Varopoulos [21,22]. Subsequently, weak spectral sets and the weak synthesis problem gained considerable attention [15,25,13,8], the more so because there are several Banach algebras for which weak synthesis holds, whereas spectral synthesis fails (see the examples given in [8, Section 1]).

In the present paper we continue our investigation [8] of weak spectral sets and weak Ditkin sets. The setting is that of a general regular and semisimple commutative Banach algebra A rather than just the Fourier algebra of a locally compact abelian group as in several other papers on the subject. Given closed subsets E and F of $\Delta(A)$, the main emphasis is on relating the weak spectral set and the weak Ditkin set properties of E, F, $E \cap F$ and $E \cup F$ and the corresponding values of ξ and η (Theorems 2.1, 2.4, 2.7 and 2.8). We also obtain results on infinite unions (Corollary 2.6 and Theorem 2.11).

One of the most interesting examples of a commutative Banach algebra for which spectral synthesis fails, but weak spectral synthesis holds, is $C^n[0,1]$, the algebra of n-times continuously differentiable functions on the interval $[0,1]$. It turns out that for each closed subset E of $[0,1] = \Delta(C^n[0,1])$, $\xi(E) = \eta(E)$ and either $\xi(E) = 1$ or $\xi(E) = n+1$. We characterize the sets E for which $\xi(E)$ attains either of the two values (Theorem 3.1).

Finally, in Section 4 we study the projective tensor product $A \widehat{\otimes} B$ of two commutative Banach algebras A and B. Under the hypothesis that $\Delta(A)$ is discrete and both A and B satisfy the weak Ditkin condition at infinity, we prove in Theorem 4.3 that if $E \subseteq \Delta(A)$ and if $F \subseteq \Delta(B)$ is a weak spectral set for B, then $E \times F \subseteq \Delta(A \widehat{\otimes} B)$ is a weak spectral set for $A \widehat{\otimes} B$, and we give upper and lower estimates for $\xi(E \times F)$. This result applies to $A(G) \widehat{\otimes} A$, where $A(G)$ is the Fourier algebra of a locally compact group G, and it yields a criterion for when weak spectral synthesis holds for $A(G) \widehat{\otimes} A$ (Theorem 4.4), thereby extending the corresponding result for $L^1(G,A) = L^1(G) \widehat{\otimes} A$, where G is a locally compact abelian group.

1. Preliminaries and some basic lemmas

Let A be a semisimple and regular commutative Banach algebra. Originally, in [24] a closed subset E of $\Delta(A)$ was defined to be a weak spectral set if every element of the quotient algebra $k(E)/\overline{j(E)}$ is nilpotent. However, as shown in [24, Theorem 1.2] and [3, footnote 7, p. 885], then there exists $n \in \mathbb{N}$ such that $a^n \in \overline{j(E)}$ for all $a \in k(E)$. One of the important features of the class of weak spectral sets is that it is closed under the formation of finite unions. Actually, for any two weak spectral sets E_1 and E_2, $\xi(E_1 \cup E_2) \leqslant \xi(E_1) + \xi(E_2)$ [24, Theorem 2.2] (see [14, Corollary 3.11] for a different approach). A closed countable union $\bigcup_{i=1}^{\infty} E_i$ of weak Ditkin sets is a weak Ditkin set provided the values $\eta(E_i)$ are bounded and A satisfies the weak Ditkin condition at infinity, that is, $\eta(\emptyset) < \infty$ [8, Proposition 1.5].