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Ranges of bimodule projections and reflexivity

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Abstract

We develop a general framework for reflexivity in dual Banach spaces, motivated by the question of when the weak* closed linear span of two reflexive masa-bimodules is automatically reflexive. We establish an affirmative answer to this question in a number of cases by examining two new classes of masa-bimodules, defined in terms of ranges of masa-bimodule projections. We give a number of corollaries of our results concerning operator and spectral synthesis, and show that the classes of masa-bimodules we study are operator synthetic if and only if they are strong operator Ditkin.

Keywords: Reflexivity; Synthesis; Schur idempotents

1. Introduction

Operator synthesis, introduced by W. Arveson [2] and subsequently developed by V.S. Shulman and L. Turowska [19,20], is an operator theoretic version of the well-known concept of spectral synthesis in Harmonic Analysis. Due to the work of W. Arveson, J. Froelich, J. Ludwig, N. Spronk and L. Turowska [2,7,22,14], it is known that the notion of spectral synthesis "embeds" into that of operator synthesis in that, for a large class of locally compact groups G, given a closed subset E of G, there is a canonical way to produce a subset E^* of the direct product

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$G \times G$, so that the set E satisfies spectral synthesis if and only if the set E^* satisfies operator synthesis. Thus, the well-known, and still open, problem of whether the union of two synthetic sets is synthetic can be viewed as a special case of the problem asking whether the union of two operator synthetic sets is operator synthetic.

The notion of operator synthesis is closely related to that of reflexivity. Recall that a subspace $\mathcal{S}$ of the space $\mathcal{B}(H_1, H_2)$ of all bounded linear operators from a Hilbert space H_1 into a Hilbert space H_2 is called *reflexive* if it coincides with its *reflexive hull* [13]

$$\operatorname{Ref} \mathcal{S} = \{T \in \mathcal{B}(H_1, H_2)\colon\ Tx \in \overline{\mathcal{S}x},\ \text{for all } x \in H_1\}.$$

Reflexive spaces are automatically closed in the weak* (and even the weak operator) topology. In the present paper we initiate the study of the following question:

Question 1.1. Given two reflexive spaces $\mathcal{S}, \mathcal{T} \subseteq \mathcal{B}(H_1, H_2)$, when is the weak* closure $\overline{\mathcal{S} + \mathcal{T}}^{w^*}$ of their sum reflexive?

Question 1.1 is closely related to the question of whether the union of two operator synthetic sets is operator synthetic. Indeed, an affirmative answer to Question 1.1, in the case $\mathcal{S}$ and $\mathcal{T}$ are bimodules over maximal abelian selfadjoint algebras (masa-bimodules for short) with operator synthetic supports, implies that the union of these supports is operator synthetic.

We obtain an affirmative answer to Question 1.1 in a number of cases. Crucial for our considerations is the class of masa-bimodules consisting of all ranges of weak* continuous bimodule projections. The latter maps have attracted considerable attention in the literature, as they are precisely the idempotent Schur multipliers (see [11]). We study a class of masa-bimodules, which we call *approximately $\mathfrak{I}$-injective* masa-bimodules, that are defined as the intersections of sequences of ranges of uniformly bounded weak* continuous masa-bimodule projections, as well as the more general class of *$\mathfrak{I}$-decomposable* masa-bimodules (Definition 2.6). Our most general result concerning Question 1.1 is that it has an affirmative answer when $\mathcal{S}$ is a reflexive masa-bimodule, while $\mathcal{T}$ is the intersection of finitely many $\mathfrak{I}$-decomposable masa-bimodules. In particular, $\overline{\mathcal{S} + \mathcal{T}}^{w^*}$ is reflexive whenever $\mathcal{S}$ is reflexive and $\mathcal{T}$ is a masa-bimodule (or a CSL algebra) of finite width. These results are given as an application of a more general result obtained in Section 2, where a new reflexive hull in the setting of dual Banach spaces is introduced and examined. We hope that this general setting may be applied in other instances as well.

Sections 4 and 5 are devoted to connections with spectral and operator synthesis. As a corollary of our results, we show that the union of an operator synthetic set and a set of finite width is operator synthetic. This extends the results of [19] and [23], where it was shown that sets of finite width are operator synthetic. We give some applications concerning unions of sets of spectral synthesis in locally compact groups. We show that the supports of the ranges of weak* continuous masa-bimodule projections are always operator synthetic. While we do not know whether the same holds for the supports of approximately $\mathfrak{I}$-injective ones, we show that these sets satisfy a weaker form of operator synthesis (see Theorem 5.2). Moreover, we show that the supports of the (more general) $\mathfrak{I}$-decomposable masa-bimodules are operator synthetic if and only if they are strong operator Ditkin. We note that it is an open question in Harmonic Analysis (resp. Operator Theory) whether every synthetic set (resp. every operator synthetic set) is necessarily Ditkin (resp. operator Ditkin).

In Section 6 we address the converse to Question 1.1, and obtain sufficient conditions which ensure the reflexivity of $\mathcal{T}$, provided $\mathcal{S}$ and $\overline{\mathcal{S} + \mathcal{T}}^{w^*}$ are both reflexive.